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Study of theory and phenomenology of some classes of family symmetry and unification models

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Abstract: We review and compare theoretically and phenomenologically a number of possible family symmetries, which when combined with unification, could be important in explaining quark, lepton and neutrino masses and mixings, providing new results in several cases. Theoretical possibilities include abelian or non-abelian, symmetric or non symmetric Yukawa matrices, Grand Unification or not. Our main focus is on anomaly-free U(1) family symmetry combined with SU(5) unification, although we also discuss other possibilities. We provide a detailed phenomenological fit of the fermion masses and mixings for several examples, and discuss the supersymmetric flavour issues in such theories, including a detailed analysis of lepton flavour violation. We show that it is not possible to quantitatively and decisively discriminate between these different theoretical possibilities at the present time.

Keywords: Supersymmetric Effective Theories.

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1. Introduction

The hierarchy of quark and charged lepton masses and the small quark mixing angles has been one of the most puzzling aspects left unresolved by the Standard Model. The recent discovery of neutrino masses and mixings has provided further clues in the search for the new physics Beyond the Standard Model which must be responsible for the pattern of fermion masses and mixing angles. One promising approach to understanding the fermion spectrum is the idea of family symmetry, and in particular the idea of a U(1) family symmetry as originally proposed by Froggatt and Nielsen [1]. Such an approach was given considerable impetus by the observation that in many string constructions additional U(1) symmetries are ubiquitous, and furthermore such a gauged broken U(1) could provide a phenomenologically viable candidate family symmetry by virtue of the Green-Schwartz anomaly cancellation mechanism [3] which provides a string solution to the no-go theorem that anomaly freedom requires such symmetries to be family independent [2]. As a result of this a considerable literature has developed in recent years based on string-inspired U(1) family symmetries [4, 5].

Many non-abelian family symmetries have also been considered, for example based on SU(3) family symmetry [6], and also textures and analyses of fermion masses have been done not using any family symmetry. At the present time some very successful approaches exist, and others that may with modification also be effective. Family symmetries can be abelian or non-abelian, they can require symmetric Yukawa matrices or not, they can be imposed with or without an associated grand unified theory, and so on. Criteria that could be used to choose among possible approaches include not only describing the quark masses and mixings, and the charged lepton masses, but also neutrino masses and mixings, supersymmetry soft breaking effects (since particularly the trilinear couplings are affected by the Yukawa couplings), how many parameters are used to describe the data, whether some results such as the Cabibbo angle are generic or fitted, and more. One of our main goals here is to look at the various possibilities systematically and see if some seem to be favoured by how well they do on a set of criteria such as the above listed ones. Presumably family symmetries originate in string theories, and are different for different string constructions that lead to a description of nature, so identifying a unique family symmetry (or a subset of possible ones) could point strongly toward a class of string theories and away from other classes. At the present time this approach is not very powerful, though it gives some interesting insights, but better analyses and additional data may improve it.

In this paper we shall consider U(1) family symmetries and unification as a viable framework for quark and lepton masses and mixing angles in the light of neutrino mass and mixing data [7], using sequential right-hand neutrino dominance [8] as a guide to constructing hierarchical neutrino mass models with bi-large mixing. As has been pointed earlier [9], models which satisfy the Gatto-Sartori-Tonin relations (GST [10])[1] require the presence of both positive and negative abelian charges. As we will discuss, the sequential dominance conditions require also the presence of both positive and negative abelian charges, and hence at least two flavon fields of equal and opposite charges. These models however result in complicated U(1) charges, on the other hand Non-GST models have a simpler charge structure and may be possible to realize in a more general context. In this work we also consider non GST cases.

We shall consider U(1) family symmetry combined with unified gauge groups based on SU(5) and SO(10), assuming a Georgi-Jarlskog relation, and also consider non-unified models without such a relation. We will present new classes of solutions to the anomaly cancellation conditions and perform phenomenological fits, and we will compare the different classes of U(1) to each other and to non-abelian family symmetry models based on SU(3) [6], by performing specific phenomenological fits to the undetermined coefficients of the operators. Finally we will consider the implications of such an approach on flavour-changing processes in the framework of supersymmetry, leaving a detailed analysis for a future reference.

The layout of the paper is as follows. In section 2 we consider the general conditions for Green-Schwartz anomaly cancellation, and move on to describe the classes of solutions, by whether they are consistent with SU(5), SO(10), Pati-Salam unification of representa-

[1] $V_{us} = |\sqrt{\frac{m_d}{m_s}} - e^{i\Phi_1}\sqrt{\frac{m_u}{m_c}}|$.

tions, generalized non-unified relations, or not at all consistent with unification. Having found these solutions, we move on in section 3 to re-parametrize in terms of differences in $\mathrm{U}(1)_F$ charges. In section 4 we consider the constraints on the Yukawa textures from requiring acceptable quark mixings and quark and lepton masses. Then in section 5, the constraints from getting acceptable neutrino masses and mixings from single right-handed neutrino dominance (SRHND) models, which are a class of see-saw models. In section 6 we construct solutions which are consistent with SU(5) unification, the Gatto-Satori-Tonin (GST) relation [10], and correct fermion masses and mixings. In section 7 we construct solutions which are consistent with SU(5) unification, correct fermion masses and mixing angles but which are not consistent with the GST relation. In section 8 we construct solutions which are not consistent with SU(5) unification. In section 9, we take some of the solutions constructed in section 6 and section 7 and fit the arbitrary $O(1)$ parameters to try to closely predict the observed fermion masses and mixing angles. Then in section 10 we briefly consider whether flavour changing processes will be dangerously high in these models, presenting two specific scenarios: a non minimal sugra possibility and a string-inspired mSUGRA-like scenario which is expected to be (or be close to) the best-case scenario for flavour-changing and for which we check explicitly $\mu \to e\gamma$ Finally, we conclude in section 11.

2. Anomaly Constraints on U(1) Family symmetries

2.1 Green-Schwartz anomaly cancellation

Consider an arbitrary U(1) symmetry which extends the Standard Model gauge group. If we were to insist that it does not contribute to mixed anomalies with the Standard Model, we would find that the generators of U(1) would be a linear combination of Weak hypercharge and $B - L$ [2]. This clearly is not useful for family symmetries, so we need to use a more sophisticated way of removing the anomalies, Green-Schwartz anomaly cancellation [3]. In this case, we can cancel the mixed $\mathrm{U}(1)-\mathrm{SU}(3)-\mathrm{SU}(3)$, $\mathrm{U}(1)-\mathrm{SU}(2)-\mathrm{SU}(2)$ and $\mathrm{U}(1)-\mathrm{U}(1)_Y-\mathrm{U}(1)_Y$ anomalies, A_3, A_2, and A_1 if they appear in the ratio:

$$A_3 : A_2 : A_1 : A_{\mathrm{U}(1)} : A_G = k_3 : k_2 : k_1 : 3k_{\mathrm{U}(1)} : 24\,, \tag{2.1}$$

where we have included the relations to the anomalies of the anomalous flavour groups $A_{\mathrm{U}(1)}$ and the gravitational anomaly; k_i are the Kac-Moody levels of the gauge groups, defined by the GUT-scale relation:

$$g_3^2 k_3 = g_2^2 k_2 = g_1^2 k_1\,. \tag{2.2}$$

If we work with a GUT that has the canonical GUT normalization, we find:

$$A_3 = A_2 = \frac{3}{5} A_1\,. \tag{2.3}$$

But we still require that the $\mathrm{U}(1)-\mathrm{U}(1)-\mathrm{U}(1)_Y$ anomaly, A_1' vanishes. Now, the anomalies are given by:

$$A_i = \frac{1}{2}\mathrm{Tr}\left[\left\{T_a^{(i)}, T_c^{(i)}\right\} T_c'\right]. \tag{2.4}$$

Field	Q_i	$\overline{U}_i$	$\overline{D}_i$	L_i	$\overline{E}_i$	$\overline{N}_i$	H_u	H_d
Charge	q_i	u_i	d_i	l_i	e_i	n_i	h_u	h_d

Table 1: Fields and family charges.

We then use the fact that $\{T_a, T_b\} = \delta_{ab}\mathbf{1}$ for SU(N) and $\{Y, Y\} = 2Y^2$ for U(1)$_Y$ to obtain:

$$A_3 = \frac{1}{2}\left[\sum_{i=1}^{3}(2q_i + u_i + d_i)\right] \tag{2.5}$$

$$A_2 = \frac{1}{2}\left[\sum_{i=1}^{3}(3q_i + l_i) + h_u + h_d\right] \tag{2.6}$$

$$\frac{3}{5}A_1 = \frac{1}{2}\left[\sum_{i=1}^{3}\left(\frac{q_i}{5} + \frac{8u_i}{5} + \frac{2}{5}d_i + \frac{3l_i}{5} + \frac{6e_i}{5}\right) + \frac{3}{5}(h_u + h_d)\right] \tag{2.7}$$

$$A_1' = \sum_{i=1}^{3}(-q_i^2 + 2u_i^2 - d_i^2 + l_i^2 - e_i^2) + (h_d^2 - h_u^2) = 0\,. \tag{2.8}$$

Since in the mixed anomalies of the U(1) group with the SM gauge group that cancel via the Green-Schwartz mechanism wherever a charge appears, it appears in a sum, we parameterize the sums as follows [12]:

$$\sum_{i=1}^{3} q_i = x + u\,, \qquad \sum_{i=1}^{3} u_i = x + 2u\,, \tag{2.9}$$

$$\sum_{i=1}^{3} d_i = y + v\,, \qquad \sum_{i=1}^{3} l_i = y\,, \tag{2.10}$$

$$\sum_{i=1}^{3} e_i = x\,, \tag{2.11}$$

$$h_u = -z\,, \qquad h_d = z + (u + v)\,. \tag{2.12}$$

Substituting eq. (2.9)–eq. (2.12) into eq. (2.5)–eq. (2.8) we find that they satisfy eq. (2.3):

$$A_3 = A_2 = \frac{3}{5}A_1 = \frac{1}{2}\left[3x + 4u + y + v\right], \tag{2.13}$$

which shows that the parameterization is consistent. However we need to find those solutions which also satisfy $A_1' = 0$. We will see how we can achieve this for different cases. Since the proposal of the GS anomaly mechanism it has been known that the easiest solution, $u = v = 0$, leads to a SU(5) or Pati-Salam group realization of mass matrices. Another possible solution is to have $u = -v \neq 0$. Both these forms admit a SUSY μ term in the tree level superpotential at the gravitational scale. However given the form of eq. (2.9)–eq. (2.12) one can try to use the flavour symmetry in order to forbid this term, allowing it just in the Kähler potential and thus invoking the Giudice-Masiero [11] mechanism in order to generate the μ of the desired phenomenological order. Therefore apart from the cases $u+v=0$ we examine plausible cases for $u \neq -v \neq 0$. Of course in the cases $u=v=0, u=-v\neq 0$ one can use another symmetry to forbid the μ term in the superpotential, however it is appealing if the flavour symmetry forbids the μ term at high scales.

2.2 Anomaly free A_1' with $u=v=0$ solutions

In this case the parameterization simplifies and in fact we can decompose the U(1) charges in flavour independent and flavour dependent parts

$$f_i = \frac{1}{3} f + f_i' \,. \tag{2.14}$$

The first term is flavour independent because it just depends on the total sum of the individual charges and the f_i' are flavour dependent charges. We can always find x and y which satisfy

$$\sum_{i=1}^{3} f_i' = 0 \,. \tag{2.15}$$

In this way A_1' can be expressed in flavour independent plus flavour dependent terms

$$A_1' = A_{1FI}' + A_{1FD}' \,. \tag{2.16}$$

Following this, with the unfortunate notation that we have a new u, completely unrelated to the u that we have already set to zero, we then have:

$$\begin{aligned} A_1' &= A_{1FI}' + A_{1FD}' \\ &= \frac{1}{3}\left[-q^2 + 2u^2 - d^2 + l^2 - e^2\right] + \sum_{i=1}^{3}(-{q_i'}^2 + 2{u_i'}^2 - {d_i'}^2 + {l_i'}^2 - {e_i'}^2) \,. \end{aligned} \tag{2.17}$$

Now it is clear that the terms in the square bracket in eq. (2.17) are family independent. It turns out that the square bracket term is automatically zero in this case, since from eqs. (2.9)–(2.11), we have: $q=u=e=x$ and $l=d=y$. Then we have to make the family dependent part (the second term in eq. (2.17)) vanish.

2.2.1 SU(5) and SO(10) type cases

One way to make the family dependent part vanish, $A_{1FD}' = 0$, is to set $l_i = d_i$ and $q_i = u_i = e_i$.[2] This condition would be automatic in SU(5), but in general such a condition on the charges does not necessarily imply a field theory SU(5) GUT to actually be present, although it may be.

Since the generic Yukawa structure is of the form:

$$Y^f \approx \begin{bmatrix} \epsilon^{|f_1+q_1+h_f|} & \epsilon^{|f_2+q_1+h_f|} & \epsilon^{|f_3+q_1+h_f|} \\ \epsilon^{|f_3+q_2+h_f|} & \epsilon^{|f_2+q_2+h_f|} & \epsilon^{|f_3+q_2+h_f|} \\ \epsilon^{|f_1+q_3+h_f|} & \epsilon^{|f_2+q_3+h_f|} & \epsilon^{|f_3+q_3+h_f|} \end{bmatrix} \tag{2.18}$$

it is clear that the SU(5) relations $d_i = l_i$, $q_i = u_i = e_i$ lead to Yukawa textures of the

[2]The reason that the charges are unprimed here is that if it is true for the primed charges, it is also true for the unprimed charges

form:

$$Y^u \approx \begin{bmatrix} \epsilon^{|2e_1-2e_3|} & \epsilon^{|e_1+e_2-2e_3|} & \epsilon^{|e_1-e_3|} \\ \epsilon^{|e_1+e_2-2e_3|} & \epsilon^{|2e_2-2e_3|} & \epsilon^{|e_2-e_3|} \\ \epsilon^{|e_1-e_3|} & \epsilon^{|e_2-e_3|} & \epsilon^{|0|} \end{bmatrix}, \tag{2.19}$$

$$Y^d \approx \begin{bmatrix} \epsilon^{|l_1+e_1+h_d|} & \epsilon^{|l_2+e_1+h_d|} & \epsilon^{|l_3+e_1+h_d|} \\ \epsilon^{|l_1+e_2+h_d|} & \epsilon^{|l_2+e_2+h_d|} & \epsilon^{|l_3+e_2+h_d|} \\ \epsilon^{|l_1+e_3+h_d|} & \epsilon^{|l_2+e_3+h_d|} & \epsilon^{|l_3+e_3+h_d|} \end{bmatrix}, \tag{2.20}$$

$$Y^e \approx Y^{d\,T}. \tag{2.21}$$

Note that the up matrix is approximately symmetric, due to the assumed SU(5) relation of charges. The reason why the textures above are approximate is that each entry in each matrix contains an undetermined order unity flavour dependent coefficient, generically denoted as $a^f_{ij} = O(1)$. We shall continue to suppress such coefficients in order to make the discussion less cumbersome, but will return to this question when we discuss the numerical fits later in the paper. We have also assumed that the up and down Yukawa matrices are described by a single expansion parameter ϵ. The possibility of having two different expansion parameters, one for the up sector and one for the down sector, will also be discussed later in the paper. In order to have an acceptable top quark mass, we have required that $h_u + 2e_3 = 0$, in which case the smallness of the bottom quark mass can be due to $h_d + e_3 + l_3 \neq 0$, and we are free to have a small $\tan\beta$, because we don't need large $\tan\beta$ to explain the ratio $\frac{m_t}{m_b}$ on its own.

Also note that, as expected from the SU(5) relation of charges, the down and electron textures are the approximate transposes of each other, $Y^d \approx (Y^e)^T$. Such a relation implies bad mass relations for between the down type quarks and charged leptons, but may be remedied by using Clebsch factors such as a Georgi-Jarlskog factor of 3 in the (2,2) position of the charged lepton Yukawa matrix.

If we were to look at the case $x = y$, then we would have a solution suggestive of unified SO(10) GUT symmetry, for which $l_i = q_i = u_i = d_i = e_i$. The same comments above also apply here, namely that such a condition on the charges, though consistent with an SO(10) GUT does not necessarily imply a field theory realization of it. The matrices eq. (2.19)–eq. (2.20) would all become equal to the same symmetric texture in eq. (2.19), in the SO(10) case that $x = y$.

2.2.2 Pati-Salam type cases

In this case, applying the Pati-Salam constraints on the charges,

$$q_i = l_i \equiv q^L_i\,, \qquad u_i = d_i = e_i = n_i \equiv q^R_i\,, \tag{2.22}$$

so we can immediately see that also for this choice of charges both the the flavour independent and dependent parts in eq. (2.17) vanishes. We have also included the right-handed neutrino charges, which do not enter into the anomaly cancellation conditions, eq. (2.5)–eq. (2.8), but with a Pati-Salam group should obey the relation of eq. (2.22). Thus in this case all the mass matrices have the form

$$Y^f = \begin{pmatrix} \epsilon^{|l_1+e_1+h_f|} & \epsilon^{|l_1+e_2+h_f|} & \epsilon^{|l_1+e_3+h_f|} \\ \epsilon^{|l_2+e_1+h_f|} & \epsilon^{|l_2+e_2+h_f|} & \epsilon^{|l_2+e_3+h_f|} \\ \epsilon^{|l_3+e_1+h_f|} & \epsilon^{|l_3+e_2+h_f|} & \epsilon^{|l_3+e_3+h_f|} \end{pmatrix} \tag{2.23}$$

for $h_f = h_u,\ h_d$. In this case we always need to satisfy $x = y$, in contrast with the generic case of SU(5) where it is not necessary $x = y$. So we can put one of the charges in terms of the other two and the parameters $x = y$

$$e_1 = x - (e_2 + e_3)\,, \qquad l_1 = x - (l_2 + l_3)\,, \quad \Rightarrow e_1 + e_2 + e_3 = l_1 + l_2 + l_3\,. \tag{2.24}$$

We have already noted that the Pati-Salam constraints on the charges imply that the anomaly A'_1 automatically vanishes. It is also a remarkable fact that the constraints in eq. (2.24) do not in practice lead to any physical constraints on the form of the Yukawa texture in eq. (2.23). In practice, assuming only that $u + v = 0$, one can start with any set of charges l_i, e_i which lead to any desired Yukawa texture, where the charges do not satisfy the anomaly free constraint in eq. (2.24). Then from any set of non-anomaly-free charges one can construct a set of anomaly-free charges which do satisfy eq. (2.24), but do not change the form of the Yukawa matrix in eq. (2.23), by simply making an equal and opposite flavour-independent shift on the charges as follows [30]: $e_i \to e_i + \Delta$, $l_i \to l_i - \Delta$. In this paper we shall not consider the Pati-Salam approach in detail.

2.3 Solutions with anomaly free A'_1 with $u + v = 0$ $(u, v \neq 0)$

In this case, we can repeat the analysis of the previous subsection, but with the general constraints. Note however, that since $u + v = 0$, $h_u = -z$ and $h_d = +z$.

Then we are left with the result that

$$A'_1 = \frac{1}{3}\left[6u^2 + 6xu + 2yu\right] - \sum_{i=1}^{3}\left(q'^{\,2}_i - 2u'^{\,2}_i + d'^{\,2}_i - l'^{\,2}_i + e'^{\,2}_i\right). \tag{2.25}$$

Note that the family independent part will vanish if

$$u = -v = -\left(x + \frac{y}{3}\right). \tag{2.26}$$

Having done this, we may substitute eq. (2.26) into eqs. (2.9)–(2.12) Then we find that:

$$\begin{aligned} &\sum_{i=1}^{3} q_i = -\frac{y}{3}\,, \qquad \sum_{i=1}^{3} u_i = -\left(x + \frac{2y}{3}\right), \\ &\sum_{i=1}^{3} d_i = x + \frac{4y}{3}\,, \qquad \sum_{i=1}^{3} l_i = y\,, \\ &\sum_{i=1}^{3} e_i = x\,. \end{aligned} \tag{2.27}$$

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2.3.1 Yukawa textures for a sample solution

At this point, we note that there will be a large number of solutions. However, one class of solutions that will easily be satisfied will be:

$$q_i = -\frac{l_i}{3}\,, \qquad u_i = -\left(\frac{2l_i}{3} + e_i\right), \qquad d_i = \frac{4l_i}{3} + e_i\,. \tag{2.28}$$

The same equation will hold for the primed charges:

$$q'_i = -\frac{l'_i}{3}\,, \qquad u'_i = -\left(\frac{2l'_i}{3} + e'_i\right), \qquad d'_i = \frac{4l'_i}{3} + e'_i\,. \tag{2.29}$$

We can now put eq. (2.29) into the anomaly, eq. (2.25). In this case we find that:

$$\begin{aligned} A'_1 &= \frac{1}{3}\left[x^2(6-6) + \frac{2}{3}y^2(1-1) + xy(4-2-2)\right] - \\ &\quad - \sum_{i=1}^{3}\left({l'_i}^2\frac{1}{9}(-1+8-16+9) + {e'_i}^2(2-1-1)\right) = 0\,. \end{aligned} \tag{2.30}$$

So we see that for this particular relation of leptonic and quark charges, we are automatically anomaly-free.

Again, we see that, just as for the $u = v = 0$ case, we can specify everything by the leptonic charges l_i and e_i. However, in this case we will get three different textures. Specifically, we will get:

$$Y^u \approx \begin{bmatrix} \epsilon^{|l_1+e_1+h_u|} & \epsilon^{|\frac{1}{3}(l_2+2l_1)+e_1+h_u|} & \epsilon^{|\frac{1}{3}(l_3+2l_1)+e_1+h_u|} \\ \epsilon^{|\frac{1}{3}(l_1+2l_2)+e_2+h_u|} & \epsilon^{|l_2+e_2+h_u|} & \epsilon^{|\frac{1}{3}(l_3+2l_2)+e_2+h_u|} \\ \epsilon^{|\frac{1}{3}(l_1+2l_2)+e_3+h_u|} & \epsilon^{|\frac{1}{3}(l_2+2l_3)+e_3+h_u|} & \epsilon^{|l_3+e_3+h_u|} \end{bmatrix} \tag{2.31}$$

$$Y^d \approx \begin{bmatrix} \epsilon^{|l_1+e_1-h_u|} & \epsilon^{|\frac{1}{3}(-l_1+4l_2)+e_2-h_u|} & \epsilon^{|\frac{1}{3}(-l_1+4l_3)+e_3-h_u|} \\ \epsilon^{|\frac{1}{3}(-l_2+4l_1)+e_1-h_u|} & \epsilon^{|l_2+e_2-h_u|} & \epsilon^{|\frac{1}{3}(-l_2+4l_3)+e_2-h_u|} \\ \epsilon^{|\frac{1}{3}(-l_1+4l_3)+e_3-h_u|} & \epsilon^{|\frac{1}{3}(-l_2+4l_3)+e_3-h_u|} & \epsilon^{|l_3+e_3-h_u|} \end{bmatrix} \tag{2.32}$$

$$Y^e \approx \begin{bmatrix} \epsilon^{|l_1+e_1-h_u|} & \epsilon^{|l_1+e_2-h_u|} & \epsilon^{|l_1+e_3-h_u|} \\ \epsilon^{|l_2+e_1-h_u|} & \epsilon^{|l_2+e_2-h_u|} & \epsilon^{|l_2+e_3-h_u|} \\ \epsilon^{|l_3+e_1-h_u|} & \epsilon^{|l_3+e_2-h_u|} & \epsilon^{|l_3+e_3-h_u|} \end{bmatrix}. \tag{2.33}$$

We note that this is a rather predictive scheme; we require that the diagonal elements are of the same order in the between the down and electron Yukawa matrices constrained by the anomalies. Also, we require (at the very least) $l_3 + e_3 + h_u = 0$ to get a correct top quark mass.

2.4 Anomaly free A'_1 with $u + v \neq 0$ solutions

In this case we can not decompose the expression of A'_1 into flavour independent and flavour dependent parts, but we can use for example the relation $\left(\sum f_i\right)^2 = \sum f_i^2 + 2(f_1(f_2+f_3) + f_2 f_3)$ such that we have

$$A'_1 = -2(4u^2 + u(v+3x+z) + v(z-y)) - 2\sum_{f=u,d,l,e,q} g_f(f_1(f_2+f_3) + f_2 f_3)\,, \tag{2.34}$$

where $g_f = 1, -2, 1, -1, 1$ respectively for $f = q, u, d, l, e$. However it is difficult to depart from here in order to find some ansatz which cancels the A_1' anomaly. Instead we can generalize the kind of relations which in the limit of $u = v = 0$ would give the SU(5) cases or the Pati-Salam cases.

2.4.1 An extended SU(5) case

Here a non-GUT case is considered, taken by generalizing the SU(5) relation between the charges. In the SU(5) case, we had $q_i = u_i = e_i$ and $d_i = l_i$. If instead we have the linear relations:

$$q_i = u_i + \alpha = e_i + \gamma\,, \qquad d_i = l_i + \beta\,. \tag{2.35}$$

From the parameterization of eqs. (2.5)–(2.8), we see that in the limit of the $u = v = 0$ we recover the SU(5) case. In agreement with the cancellation of anomalies then one should have

$$q_i = u_i - \frac{u}{3} = e_i + \frac{u}{3}\,, \qquad d_i = l_i + \frac{v}{3}\,. \tag{2.36}$$

In the expression of the A_1' anomaly, as given in eq. (2.8), the sums of squared charges cancel and we can write it just in terms of sum of charges, which we have parameterized in terms of u, v, x, y,

$$A_1' = -10\frac{u^2}{3} - \frac{2}{3}v^2 + 2u(x+v) + 2y\frac{v}{3} - 2z(u+v) = 0\,. \tag{2.37}$$

Thus we need to satisfy this equation in order to have anomaly free solutions. Requiring the condition of $O(1)$ top coupling we have

$$\begin{aligned} h_u &= -z = -2e_3 - u\,, \\ h_d &= 2u + v + 2e_3\,, \\ \mathcal{C}(Y^u_{ij}) &= |e_i + e_j - 2e_3|\,, \\ \mathcal{C}(Y^d_{ij}) &= \left|e_i + l_j + 2e_3 + \frac{7u}{3} + \frac{4v}{3}\right|, \\ \mathcal{C}(Y^e_{ij}) &= |l_i + e_j + 2e_3 + 2u + v|\,, \end{aligned} \tag{2.38}$$

where $\mathcal{C}(Y^u_{ij})$ denotes the power of ϵ for the (i, j) element of the correspondent Yukawa matrix. Note that although we did not begin with an *a priori* condition of having Y^u symmetric, the requirement of the $O(1)$ top coupling cancels the parameter u in all the entries of Y^u and so we end up with a symmetric matrix.

2.4.2 An extended Pati-Salam case

Following the extended SU(5) case, we look for solutions which in the $u = v = 0$ limit reproduce the Pati-Salam case, so we should have the relations

$$q_i = l_i + \alpha\,, \qquad u_i = d_i + \beta\,. \tag{2.39}$$

Also e_i and n_i need to be related to u_i by a constant, as in eq. (2.39). In these case in order to satisfy the G-S anomaly conditions we need

$$q_i = l_i + \frac{u + (x-y)}{3}\,, \qquad u_i = e_i + \frac{2u}{3}\,, \qquad d_i = e_i + \frac{v + (y-x)}{3}\,. \tag{2.40}$$

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Thus the expression for the A_1' anomaly is

$$A_1' = -\frac{2}{9}\left[8u^2+4v^2+u(9v+11x-2y)+2(x-y)^2-v(2x+y)\right]- \\ -2z(u+v)\,, \qquad (2.41)$$

and finally requiring the condition of $O(1)$ top Yukawa coupling we have

$$\begin{aligned} h_u &= -z = -\left(l_3+e_3+u+\frac{x-y}{3}\right), \\ h_d &= l_3+e_3+2u+v+\frac{x-y}{3}, \\ \mathcal{C}(Y^u_{ij}) &= |l_i-l_3+e_j-e_3|\,, \\ \mathcal{C}(Y^d_{ij}) &= \left|l_i+e_j+l_3+e_3+\frac{4v+7u+(x-y)}{3}+\frac{4v}{3}\right|, \\ \mathcal{C}(Y^e_{ij}) &= |l_i+e_j+2e_3+2u+v|\,. \end{aligned} \qquad (2.42)$$

3. A useful phenomenological parameterization

So far we have discussed the anomaly cancellation conditions in U(1) family symmetry models, and some of the possible solutions to these conditions, including some new solutions not previously discussed in the literature. It turns out however that the anomaly free charges themselves do not provide the most convenient parameters for discussing the phenomenological constraints on the Yukawa matrices arising from the quark and lepton spectrum. It is more convenient to introduce a new parameterization for the Yukawa matrices as follows:

$$Y^f \approx \begin{pmatrix} \epsilon^{|s'_f+r'_f+k_f|} & \epsilon^{|s'_f+r_f+k_f|} & \epsilon^{|s'_f+k_f|} \\ \epsilon^{|s_f+r'_f+k_f|} & \epsilon^{|s_f+r_f+k_f|} & \epsilon^{|s_f+k_f|} \\ \epsilon^{|r'_f+k_f|} & \epsilon^{|r_f+k_f|} & \epsilon^{|k_f|} \end{pmatrix}, \qquad (3.1)$$

where $f = u, d, e, \nu$, and we have introduced the parameters $r_f, r'_f, s_f, s'_f, k_f$ which are defined in terms of the charges in table 1 as:

$$\begin{aligned} r_f &= f_2-f_3 & r'_f &= f_1-f_3 & k_u &= q_3+u_3+h_u \\ s_{u,d} &= q_2-q_3 & s'_{u,d} &= q_1-q_3 & k_d &= q_3+d_3+h_d \\ s_{e,\nu} &= l_2-l_3 & s'_{e,\nu} &= l_1-l_3 & k_e &= l_3+e_3+h_d \\ & & & & k_\nu &= l_3+n_3+h_u\,. \end{aligned} \qquad (3.2)$$

In order to get an acceptable top quark mass, we require that $k_u = 0$. Note that the parametrization above is completely general, there is no information loss from the form of eq. (2.18), and thus far we have not imposed any constraints on the charges arising from either anomaly cancellation or from GUTs. We now consider the simplifications which arise in the new parametrization when the charges are constrained by considerations of anomaly cancellation and GUTs, as discussed in the previous section.

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Simplification in SU(5) **type case.** Consider the case where the family charges are consistent with the representations in an SU(5) GUT, $d_i = l_i$, and $q_i = u_i = e_i$:

$$\begin{aligned} k_e &= k_d \qquad & s_{u,d} &= r_{u,e} \qquad & s'_{u,d} &= r'_{u,e} \\ s_{e,\nu} &= r_d \qquad & s'_{e,\nu} &= r'_d\,. \end{aligned} \tag{3.3}$$

In this case, all of the parameters can be expressed purely in terms of the lepton charges:

$$\begin{aligned} s_{u,d} &= r_{u,e} = e_2 - e_3 \qquad & s'_{u,d} &= r'_{u,e} = e_1 - e_3 \\ s_{e,\nu} &= r_d = l_2 - l_3 \qquad & s'_{e,\nu} &= r'_d = l_1 - l_3\,. \end{aligned} \tag{3.4}$$

Note that this leads directly to the fact that $Y^e \approx (Y^d)^T$. The equality is broken by the arbitrary $O(1)$ coefficients. As discussed, the SU(5) charge conditions are sufficient to guarantee anomaly cancellation for the case $u = v = 0$.

Simplification in the extended SU(5) **case.** In the case $u + v \neq 0$, anomalies can again be cancelled by assuming the charge conditions in eq. (2.35). If we take eq. (2.35), we can again simplify eq. (3.2). In this case we find:

$$\begin{aligned} s_{u,d} &= r_{u,e} \qquad & s'_{u,d} &= r'_{u,e} \\ s_{e,\nu} &= r_d \qquad & s'_{e,\nu} &= r'_d\,. \end{aligned} \tag{3.5}$$

In this case we have that the texture of Y^e can be attained from Y^d by replacing k_d with k_e and then transposing.

Simplification in the Pati-Salam case. In the case of having charge relations consistent with a Pati-Salam theory, $q_i = l_i$ and $u_i = d_i = e_i = n_i$, we can simplify:

$$\begin{aligned} k_e &= k_d \qquad & s_{u,d} &= s_{e,\nu} \qquad & s'_{u,d} &= s'_{e,\nu} \\ k_u &= k_\nu \qquad & r_u &= r_d = r_e = r_\nu \qquad & r'_u &= r'_d = r'_e = r'_\nu\,. \end{aligned} \tag{3.6}$$

4. Quark masses and mixings in SU(5)

In this section we shall provide some constraints on the phenomenological parameters introduced in the last section, arising from the quark masses and mixings, assuming the simplification in the SU(5) type case mentioned above. In SU(5) eqs. (3.1), (3.3) imply the quark Yukawa matrices are explicitly of the form:

$$Y^u \approx \begin{pmatrix} \varepsilon^{|2s'|} & \varepsilon^{|s'+s|} & \varepsilon^{|s'|} \\ \varepsilon^{|s'+s|} & \varepsilon^{|2s|} & \varepsilon^{|s|} \\ \varepsilon^{|s'|} & \varepsilon^{|s|} & 1 \end{pmatrix}, \qquad Y^d \approx \begin{pmatrix} \varepsilon^{|s'+r'_d+k_d|} & \varepsilon^{|s'+r_d+k_d|} & \varepsilon^{|s'+k_d|} \\ \varepsilon^{|s+r'_d+k_d|} & \varepsilon^{|s+r_d+k_d|} & \varepsilon^{|s+k_d|} \\ \varepsilon^{|r'_d+k_d|} & \varepsilon^{|r_d+k_d|} & \varepsilon^{|k_d|} \end{pmatrix}, \tag{4.1}$$

where we have written $s = s_{u,d} = r_{u,e}$, $s' = s'_{u,d} = r'_{u,e}$.[3] Note that we are assuming a single expansion parameter ε, and are suppressing $O(1)$ coefficients. Clebsch factors are also not considered, and only leading order operators are discussed.

[3]Note that the extended SU(5) anomaly free solutions examined in section 2.4.1 leave the parameters s, s', r_d, r'_d, k_d invariant, as is clear by comparing eqs.(3.3) and (3.5). Hence the results in this section for the quark sector apply not only to the SU(5) type case but also the extended SU(5) anomaly free cases.

In order to determine the possible solutions for s, s', r_d, r'_d and k_d which successfully reproduce quark masses and mixings one can numerically diagonalize Yukawa matrices and obtain the CKM matrix. However, in order to understand the behaviour of this structure it is quite useful to use the technique of diagonalization by blocks in the $(2,3)$, $(1,3)$ and $(1,2)$ sectors.[4] The results are presented in the next subsections.

4.1 Quark Masses

Barring accidental cancellations the down quark Yukawa matrix Y^d may be diagonalized, leading to the following eigenvalues:

$$
\begin{aligned}
y_1 &\approx a_{11}\varepsilon^{|s'+r'+k|} - \frac{(a_{31}\varepsilon^{|r'+k|} + a_{23}a_{21}\varepsilon^{|s+k|+|s+r'+k|-|k|}e^{2i(\beta_2^L-\beta_1^L)})}{c_{23}^R(\varepsilon^{|k|} + a_{32}^2\varepsilon^{2|r+k|-|k|}e^{-2i(\beta_2^R-\beta_1^R)})} \times \\
&\quad \times (a_{13}\varepsilon^{|s'+k|} + a_{23}a_{12}\varepsilon^{|r+k|+|s'+r+k|-|k|}e^{-2i(\beta_2^R-\beta_1^R)}) - \\
&\quad \frac{-(a_{12}\varepsilon^{|s'+r+k|} - a_{32}a_{13}\varepsilon^{|r+k|+|s'+k|-|k|})(a_{21}\varepsilon^{|s+r'+k|} - a_{23}a_{31}\varepsilon^{|s+k|+|r'+k|-|k|})}{(a_{22}\varepsilon^{|s+r+k|} - a_{23}a_{32}\varepsilon^{|s+k|+|r+k|-|k|})e^{-i(\beta_3^L-\beta_3^R)}}, \\
y_2 &\approx c_{23}^R\left(a_{22}\varepsilon^{|s+r+k|} - a_{23}a_{32}\varepsilon^{|r+k|+|s+k|-|k|}\right)e^{2i(\beta_2^L-\beta_2^R)}, \\
y_3 &\approx c_{23}^R\left(\varepsilon^{|k|} + a_{32}^2\varepsilon^{2|r+k|-|k|}e^{2i(\beta_1^R-\beta_2^R)}\right)e^{i(\beta_1^L-\beta_1^R)},
\end{aligned}
\tag{4.2}
$$

where we have suppressed the index d in order to make clearer the notation and re-scaled all the (complex) coefficients by $1/a_{33}$, so that instead of having a_{33} we have 1. Note that the down quark masses are given by: $m_i^d = y_i^d v_d/\sqrt{2}$. Analogous results also apply to the up quark sector, with the replacements $r \to s$, $r' \to s'$, $k \to 0$. The phases β_i^L correspond to the diagonalization matrices of the Yukawa matrices, whose notation is given in appendix A.

It is important to remark that in the case of positive charges all the elements of the first row of the Yukawa matrix contribute at the same order, $s' + r' + k$, to their correspondent lightest eigenvalue, so in these cases it is not possible to have the Gatto-Sartori-Tonin (GST) relation. However in the cases of having s and s' (analogous for r and r') with different sign, as in the example of eq. (4.9), we can have a cancellation in powers of ε to the contribution to y_1 coming from the diagonalization in the $(1,2)$ sector, which is the third term in the expression for y_1 in eq. (4.2). On the other hand we can have an enhancement in the power of ε of the contributions from the $(1,1)$ entry and the rotation in the $(1,3)$ sectors, which correspond to the first and second term of y_1, respectively, in eq. (4.2). This together with the condition $\mathcal{C}(Y_{21}) = \mathcal{C}(Y_{12})$ are the requirements to achieve the GST relation. We will present examples satisfying and not satisfying the GST relation.

We remark here the constraints from the bottom mass are

$$
m_b \tan\beta = \varepsilon^{|k_d|}m_t\,, \qquad k_d = q_3 + d_3 + h_d \tag{4.3}
$$

since $m_t = O(\langle H_u\rangle)$ and $\tan\beta = \langle H_u\rangle/\langle H_d\rangle$. Thus in terms of charges we have $h_u = -(q_3 + u_3)$ and $h_d = q_3 + u_3$, for $u = v = 0$, $k = 2q_3 + d_3 + u_3$.

[4]This only works if there is an appropriate hierarchy among the elements

4.2 Quark mixings

We can also obtain the mixing angles in this approximation and compare to the required experimental values (see appendix B). The mixing angles in the down sector, again dropping flavour indices, are as follows:

$$
\begin{aligned}
t^L_{23} &= e^{i(\beta_2^L-\beta_1^L)} a_{23}\varepsilon^{|s+k|-|k|} + a_{23}a_{22}\varepsilon^{|s+r+k|+|s+k|-2|k|} e^{i\xi_L} \\
t^R_{23} &= e^{i(\beta_2^R-\beta_1^R)} a_{32}\varepsilon^{|r+k|-|k|} + a_{23}a_{22}\varepsilon^{|s+r+k|+|s+k|-2|k|} e^{i\xi_R} \\
t^L_{13} &= \frac{a_{13}\varepsilon^{|s'+k|} + a_{32}a_{12}\varepsilon^{|r+k|+|s'+r+k|-|k|}e^{-i2(\beta_2^R-\beta_1^R)}}{\left(\varepsilon^{|k|}+a_{32}^2\varepsilon^{2|r+k|-|k|}e^{2i(\beta_1^R-\beta_2^R)}\right)e^{i\beta_1^L}} \\
t^R_{13} &= \frac{a_{31}\varepsilon^{|r'+k|} + a_{23}a_{21}\varepsilon^{|s+k|+|s+r'+k|-|k|}e^{2i(\beta_2^L-\beta_1^L)}}{\left(\varepsilon^{|k|}+a_{32}^2\varepsilon^{2|r+k|-|k|}e^{2i(\beta_1^R-\beta_2^R)}\right)e^{-i\beta_1^R}}\sqrt{1+|a_{32}^2|\varepsilon^{2|r+k|-2|k|}} \\
t^L_{12} &= \frac{\left(a_{12}\varepsilon^{|s'+r+k|} - a_{32}a_{13}\varepsilon^{|r+k|+|s'+k|-|k|}\right) e^{-i(\beta_3^R+\beta_2^L)}}{\left(a_{22}\varepsilon^{|s+r+k|} - a_{23}a_{32}\varepsilon^{|s+k|+|r+k|-|k|}\right)} \\
t^R_{12} &= \frac{\left(a_{21}\varepsilon^{|s+r'+k|} - a_{23}a_{31}\varepsilon^{|s+k|+|r'+k|-|k|}\right) e^{i(\beta_3^L+\beta_2^R)}}{\left(a_{22}\varepsilon^{|s+r+k|} - a_{23}a_{32}\varepsilon^{|s+k|+|r+k|-|k|}\right)} \\
\xi_L &= -(\beta_2^L-\beta_1^L) - 2(\beta_2^R-\beta_1^R) \qquad \xi_R = -(\beta_2^R-\beta_1^R) - 2(\beta_2^L-\beta_1^L),. \qquad (4.4)
\end{aligned}
$$

Analogous results also apply to the up quark sector, with the replacements $r_d \to s$, $r'_d \to s'$, $k_d \to 0$. Note that in the case of positive s, s', r, r' and k, the angles t^L_{12} and t^L_{23}, of the left sector do not depend on r_d, r'_d, so they are equal, at first approximation, for the up and down sectors. Having the tangent of the angles expressed in terms of the Yukawa elements we can see directly their contributions to the CKM elements ($V_{\mathrm{CKM}} = L^u L^{d\dagger}$ in the notation of appendix A)

$$
\begin{aligned}
\frac{|V_{ub}|}{|V_{cb}|} &= \frac{|s^u_{12}s^Q_{23} - s^Q_{13}e^{i(\Phi_1-\Phi_2)}|}{|s^Q_{23}|} \approx 0.09 \sim (\lambda^2, \lambda) \\
\frac{|V_{td}|}{|V_{ts}|} &= \frac{|s^d_{12}s^Q_{23} - s^Q_{13}e^{i(\Phi_2)}|}{|s^Q_{23}|} \sim \lambda \\
|V_{us}| &= |s^d_{12} - s^u_{12}e^{i\Phi_1}| = \lambda \approx 0.224 \\
\mathrm{Im}\{J\} &= s^Q_{23}(s^Q_{23}s^d_{12}s^u_{12}\sin(\Phi_1) - s^Q_{13}(s^d_{12}\sin(\Phi_2)) - s^u_{12}\sin(\Phi_2-\Phi_1))\,, \qquad (4.5)
\end{aligned}
$$

with $s^Q_{ij} = |s^d_{ij} - e^{i\Phi_{X_{ij}}} s^u_{ij}|$. The phases Φ_1, Φ_2 and $\Phi_{X_{ij}}$ depending on the contributions that the mixing angles receive from the different elements of the Yukawa matrix and have a different expression in terms of the phases of the Yukaw matrix for different cases. For example when the elements $(1,2)$ and $(1,3)$ are of the same order and the right handed mixing angle in the $(2,3)$ sector is large, the Φ_2 phase will be

$$\Phi_2 = \mathrm{Arg}\left[\frac{Y^d_{12} + Y^d_{13}t^R_{23}}{Y^d_{33} + Y^d_{23}t^R_{23}}\right]. \qquad (4.6)$$

As we can see from the expressions in eq. (4.5) involving Φ_1, this can be associated to the U sector. When all the signalization angles in this sector are small, then this phase takes

U(1) relations	Constraint	Reason	U(1) relations	Constraint	Reason
$\varepsilon^{\|s+k_d\|-\|k_d\|}$	$\sim \lambda^2$	s_{23}^Q	$\varepsilon^{\|3q_3+d_3\|}$	$\sim (1, \lambda^3)$	m_b
$\varepsilon^{\|s'+k_d\|-\|k_d\|}$	$\gtrsim \lambda^3$	s_{13}^Q	$\varepsilon^{\|s+r_d+k_d\|-\|k_d\|}$	$\sim (\lambda^2,\ \lambda^3)$	$\frac{m_s}{m_b}$
$\varepsilon^{\|s'+r_d+k_d\|-\|s+r_d+k_d\|}$	$\sim \lambda$	s_{12}^Q	$\varepsilon^{\|s'+r'_d+k_d\|-\|k_d\|}$	$\sim (\lambda^4,\ \lambda^5)$	$\frac{m_d}{m_s}$
			$\varepsilon^{\|2s+k_d\|-\|k_d\|}$	$\sim \lambda^4$	$\frac{m_c}{m_t}$
			$\varepsilon^{\|2s'+k_d\|-\|k_d\|}$	$\geq \lambda^6$	$\frac{m_u}{m_c}$

Table 2: Constraints on the parameters s, s', r_d, r'_d and k_d from quark mixing angles and mass ratios. For the mixing angles we need to satisfy the conditions for up or down sector, where the analogous conditions for the up sector are obtained by making the replacements $r_d \to s$, $r'_d \to s'$, $k_d \to 0$. They do not need to be satisfied for both as long as for the sector in which they are not satisfied they do not give a bigger contribution than the indicated power.

the form

$$\Phi_1 = \phi_{12}^u - \phi_{22}^u\,, \tag{4.7}$$

where ϕ_{12} and ϕ_{22} are the phases of the Y_{12}^u and Y_{22}^u elements. Finally the phases Φ_{ij}, which appear in s_{ij}^Q, can be associated either with the U or with the D sector.

With the requirements of table 2 and the values of quark masses in appendix B, we can identify the viable solutions in the quark sector. One solution which has been widely explored is the up-down symmetric case for which we have $x = y$ thus, $f_i = q_i = u_i = e_i = d_i = l_i$. In this case $h_u = -2e_3 = -h_d$ so $k_u = 0$, $k_d = k_l = 4e_3$, but in this case we need two expansion parameters ε_u and ε_d to reproduce appropriate mass ratios and mixings, thus we have

$$Y^f = \begin{bmatrix} \varepsilon_f^{|2s'+k_f|} & \varepsilon_f^{|s+s'+k_f|} & \varepsilon_f^{|s'+k_f|} \\ \varepsilon_f^{|s+s'+k_f|} & \varepsilon_f^{|2s+k_f|} & \varepsilon_f^{|s+k_f|} \\ \varepsilon_f^{|s'+k_f|} & \varepsilon_f^{|s+k_f|} & \varepsilon^{|k_f|} \end{bmatrix}. \tag{4.8}$$

We can think of fixing $s + s'$, and then check for which choice of s we have appropriate phenomenological solutions. For example if we take $s + s' = \pm 3$ and $e_3 = 0$ ($k_f = 0$, $\forall f$) we have

$$Y^f = \begin{bmatrix} \varepsilon_f^{|6-2f_2|} & \varepsilon_f^{|3|} & \varepsilon_f^{|3-f_2|} \\ \varepsilon_f^{|3|} & \varepsilon_f^{|2f_2|} & \varepsilon_f^{|f_2|} \\ \varepsilon_f^{|3-f_2|} & \varepsilon_f^{|f_2|} & 1 \end{bmatrix}. \tag{4.9}$$

The viable phenomenological fit for the case of quarks is for $f_2 = -1$ and $f_1 = 4$ or $f_2 = 1$ and $f_1 = -4$ [17]. In this case we have then $x = y = \pm 3$ respectively.

5. Neutrino masses and mixings in SRHND

In this section we apply the requirements of getting acceptable neutrino masses and mixings by using a class seesaw model where $l_2 = l_3$. These are a subset of a class of seesaw models called single right-handed neutrino dominance (SRHND) or sequential dominance [8]. This additional constraint $l_2 = l_3$ will henceforth be applied in obtaining phenomenological solutions in the lepton sector.

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Apart from the obvious benefit of considering the neutrino sector, it will turn out that the neutrino sector will constrain the absolute values of the charges under the U(1) family symmetry, (not the charge differences,) due to the Majorana nature of neutrinos. This is due to the relations between the charges imposed by the relevant GUT constraints, or the extended GUT constraints, eq. (2.36) for the extended SU(5) solution of section 2.4.1 and eq. (2.40) for the extended Pati-Salam solution of section 2.4.2. For example the additional constraint $l_2 = l_3$ implies immediately

$$r_d = s_{e,\nu} = l_2 - l_3 = 0\,, \tag{5.1}$$

in the SU(5) type cases from eq. (3.4).

Here we would like to study the cases for which large mixing angles in the atmospheric sector and the neutrino sector can be explained naturally in terms of the parameters of the U(1) class of symmetries that we have constructed in the previous sections, under the framework of the type I see-saw mechanism together with the scenario of the single right handed neutrino dominance (SRHND). We refer the reader for a review of this scenario to [8]. Here we make a brief summary of the results and apply them to the present cases. In the type I see-saw the mass matrix of the low energy neutrinos is given by $m_{LL} \approx v_u^2 Y^\nu M_R^{-1} Y^{\nu T}$, where Y^ν is the Dirac matrix for neutrinos and M_R is the Majorana matrix for right-handed neutrinos. If we have three right handed neutrinos, M_1, M_2 and M_3, then for the right handed neutrino mass, in terms of U(1) charges we have:

$$Y^\nu = \begin{bmatrix} \epsilon^{|l_1+n_1+h_u|} & \epsilon^{|l_1+n_2+h_u|} & \epsilon^{|l_1+n_3+h_u|} \\ \epsilon^{|l_2+n_1+h_u|} & \epsilon^{|l_2+n_2+h_u|} & \epsilon^{|l_2+n_3+h_u|} \\ \epsilon^{|l_3+n_1+h_u|} & \epsilon^{|l_3+n_2+h_u|} & \epsilon^{|l_3+n_3+h_u|} \end{bmatrix} \tag{5.2}$$

$$M_{RR} = \begin{bmatrix} \varepsilon^{|2n_1+\sigma|} & \varepsilon^{|n_1+n_2+\sigma|} & \varepsilon^{|n_1+n_3+\sigma|} \\ \varepsilon^{|n_1+n_2+\sigma|} & \varepsilon^{|2n_2+\sigma|} & \varepsilon^{|n_2+n_3+\sigma|} \\ \varepsilon^{|n_1+n_3+\sigma|} & \varepsilon^{|n_2+n_3+\sigma|} & \varepsilon^{|2n_3+\sigma|} \end{bmatrix} \langle\Sigma\rangle\,, \tag{5.3}$$

where the charges n_i are the U(1) charges of the right handed neutrinos, ν_{Ri} and σ is the U(1) charge of the field Σ giving Majorana masses to the right handed neutrinos. These charges are not constrained by the anomaly cancellation conditions eq. (2.9)–eq. (2.12) of section 2, at least in the SU(5) case, which gives some freedom in order to find appropriate solutions giving two large mixing angles and one small mixing angle for neutrinos. We expect Σ to be of order the scale at which the U(1) symmetry is broken, for example at $M_P = M_{\text{Planck}}$, or some other fundamental scale, such as the Grand Unification scale, M_G, for the solutions with an underlying GUT theory.

Here we restrict ourselves to the cases in which eq. (5.3) can be considered as diagonal, $M_R \approx \mathrm{diag}\{\mathrm{M_1}, \mathrm{M_2}, \mathrm{M_3}\}$, for which we need in the $(2,3)$ block

$$\begin{aligned} |n_3 + n_2 + \sigma| &> \min\{|2n_3+\sigma|, |2n_2+\sigma|\}, \\ 2|n_3+n_2+\sigma| &\geq |2n_3+\sigma| + |2n_2+\sigma|\,. \end{aligned} \tag{5.4}$$

The conditions in the $(1,2)$ block are analogous to the $(2,3)$ and also we need

$$|n_1 + n_3 + \sigma| > \max\{|2n_2+\sigma|, |2n_3+\sigma|\}\,. \tag{5.5}$$

Now, there are two cases that we can consider here, which correspond to selecting which of the neutrinos will dominate, M_1 or M_3. For the later case the SRHND conditions are

$$\frac{|Y^\nu_{i3}Y^\nu_{j3}|}{|M_3|} \gg \frac{|Y^\nu_{i2}Y^\nu_{j2}|}{|M_2|} \gg \frac{|Y^{\nu 2}_{31}, Y^{\nu 2}_{21}, Y^\nu_{21}, Y^\nu_{31}|}{|M_1|}\,; \quad i,j = 1,2,3\,. \tag{5.6}$$

For the case in which M_1 dominates we just have to interchange the indices 1 and 3 in the neutrino Yukawa terms.

For the case in which M_3 dominates, at first order approximation, we have the following expressions for the neutrino mixings [8],

$$t^\nu_{23} = \frac{Y^\nu_{23}}{Y^\nu_{33}}\,, \tag{5.7}$$

$$t^\nu_{13} = \frac{Y^\nu_{13}}{\sqrt{Y^{\nu 2}_{33}+Y^{\nu 2}_{23}}} + \frac{M_3}{M_2}\frac{Y^\nu_{12}(s_{23}Y^\nu_{22}+c_{23}Y^\nu_{32})}{\sqrt{Y^{\nu 2}_{33}+Y^{\nu 2}_{23}}}\,, \tag{5.8}$$

$$t^\nu_{12} = \frac{Y^\nu_{12}(Y^{\nu 2}_{33}+Y^{\nu 2}_{23}) - Y^\nu_{13}(Y^\nu_{33}Y^\nu_{32}-Y^\nu_{22}Y^\nu_{23})}{(Y^\nu_{33}Y^\nu_{33}-Y^\nu_{32}Y^\nu_{23})\sqrt{Y^{\nu 2}_{33}+Y^{\nu 2}_{23}+Y^{\nu 2}_{13}}} \approx \frac{Y^\nu_{12}}{c_{23}Y^\nu_{22}-s_{23}Y^\nu_{32}}\,. \tag{5.9}$$

In terms of the abelian charges the Yukawa elements are

$$Y^\nu_{ij} = \varepsilon^{|l_i+n_j+h_u|} \equiv \varepsilon^{|l'_i+n_j|}, \quad l'_i \equiv l_i + h_u = l_i - 2e_3\,, \tag{5.10}$$

where we have defined primed lepton doublet charges which absorb the Higgs charge, as shown. We can work here in terms of the primed charges, once they are fixed we can determine the original abelian charges (unprimed). The approximation in eq. (5.9) corresponds to the case in which we have enough suppression of the second term in the expression for $t\nu_{12}$. In eq. (5.8) the second term can be neglected sometimes, depending on the ratio M_3/M_2. The heaviest low energy neutrino masses are given by

$$m_{\nu_3} = \frac{a^{\nu 2}_3 \varepsilon^{2|l'_2+n_3|}v^2}{M_3}\,, \qquad m_{\nu_2} = \frac{a^{\nu 2}_2 \varepsilon^{2|l'_2+n_2|}v^2}{M_2}\,, \tag{5.11}$$

where we have written $a^{\nu 2}_3\varepsilon^{2|l'_2+n_3|} = Y^{\nu 2}_{33}+Y^{\nu 2}_{23}$ and $a^{\nu 2}_2\varepsilon^{2|l'_2+n_2|} = (c_{23}Y^\nu_{22}-s_{23}Y^\nu_{32})^2$. Thus the ratio of the differences of the solar to atmospheric neutrino can be written as

$$\frac{m_{\nu_2}}{m_{\nu_3}} \approx \frac{M_3}{M_2}\frac{c^2_{23}}{c^2_{12}}\frac{(Y^\nu_{22}-Y^\nu_{32}t\nu_{23})^2}{Y^{\nu 2}_{33}+Y^{\nu 2}_{23}} \sim \varepsilon^{p_2-p_3}\,, \tag{5.12}$$

where

$$p_k = |2l'_2+n_k| - |2n_k+\sigma|\,, \qquad \text{for } k=2,\ 3\,. \tag{5.13}$$

Note that p_k is then defined such that

$$m_{\nu_k} \approx \frac{v^2}{\langle\Sigma\rangle}\epsilon^{p_k}\,. \tag{5.14}$$

6. SU(5) solutions satisfying the GST relation

In this section we shall continue to focus on the case of SU(5), where the quark Yukawa matrices take the form of eq. (4.1), where, motivated by large atmospheric neutrino mixing, we shall assume $r_d = 0$ from eq. (5.1) The purpose of this section is to show how the GST relation can emerge from SU(5), by imposing additional constraints on the parameters.[5]

6.1 The quark sector

We have already seen that the GST relation can be achieved in the u sector, mainly by allowing the parameters s and s' to have different signs. In the down sector to satisfy GST we additionally require:

$$\begin{aligned} |k_d + r'_d + s| &= |k_d + s'| \\ |k_d + r'_d + s'| - |k_d| &> |k_d + r'_d + s| + |k_d + s'| - |k_d + s| \\ |r'_d + k_d| &> |k_d| \,. \end{aligned} \tag{6.1}$$

The first of these equations ensures the equality of the order of the elements $(1,2)$ and $(2,1)$ of the Y^d matrix. The second equation ensures that the element $(1,1)$ is suppressed enough with respect to the contribution from the signalization of the $(1,2)$ block. This last condition is usually satisfied whenever $|k_d + r'_d + s'| > |k_d + r'_d + s|$ is satisfied. Finally the third condition ensures a small right-handed mixing for d-quarks and a small left-handed mixing for charged leptons. Now in order to satisfy the relations

$$s^u_{12} = \sqrt{\frac{m_u}{m_c}} \approx \lambda^2 \,, \qquad s^d_{12} = \sqrt{\frac{m_d}{m_s}} \approx \lambda \,, \tag{6.2}$$

we need a structure of matrices, in terms of just one expansion parameter $\varepsilon = O(\lambda)$, such as

$$Y^u = \begin{bmatrix} \ldots & \varepsilon^6 & \ldots \\ \varepsilon^6 & \varepsilon^4 & \varepsilon^2 \\ \ldots & \varepsilon^2 & 1 \end{bmatrix}, \qquad Y^d = \begin{bmatrix} \ldots & \varepsilon^5 & \varepsilon^5 \\ \varepsilon^5 & \varepsilon^4 & \varepsilon^4 \\ \ldots & \varepsilon^2 & \varepsilon^2 \end{bmatrix}, \tag{6.3}$$

for which we have

$$\begin{aligned} &s^u_{12} \approx \varepsilon^2 \,, \qquad s^d_{12} \approx \varepsilon \,, \qquad s^d_{23} \approx \varepsilon^2 \,, \qquad s^d_{13} \approx \varepsilon^3 \,, \\ &\frac{m_c}{m_t} \approx \varepsilon^4 \,, \qquad \frac{m_s}{m_b} \approx \varepsilon^2 \,, \qquad \frac{m_b}{m_t} \approx \varepsilon^2 \,, \end{aligned} \tag{6.4}$$

in agreement with observed values for quark masses and mixings for $\varepsilon = \lambda$.

Now we can proceed as in the example of eq. (4.9) where $s' + s$ is fixed to be ± 3. In this case we see that we can have plausible solutions in the up sector by allowing half integer solutions

$$|s' + s| = \frac{13}{2} \,, \qquad 6 \,, \qquad \frac{11}{2} \,. \tag{6.5}$$

We will refer to these solutions as Solution 1, 2 and 3 respectively. Note that only the charge differences are constrained here, the actual charges are not.

[5] Note that results in section 6 and in section 7 apply to both SU(5) type and extended SU(5) models, as discussed above.

Here we remark that the power of the expansion parameter ε, in units of the flavon field θ, accompanying the Yukawa terms has to be positive and integer. In what follows we find that we can only have solutions with fractionary charges for the combinations of charges $|f_1 + q_1 + h_f|$ of fields in the Yukawa terms such that we must allow for θ to have a fractionary charge q_θ and hence the net charge $|q_\theta||f_1 + q_1 + h_f|$ is an integer.

Solution 1, $|s + s'| = 13/2$,

$$Y^u = \begin{bmatrix} \varepsilon^{35/2} & \varepsilon^{13/2} & \varepsilon^{35/4} \\ \varepsilon^{13/2} & \varepsilon^{9/2} & \varepsilon^{9/4} \\ \varepsilon^{35/4} & \varepsilon^{9/4} & 1 \end{bmatrix}, \qquad Y^d = \begin{bmatrix} \varepsilon^{69/4} & \varepsilon^{25/4} & \varepsilon^{25/4} \\ \varepsilon^{25/4} & \varepsilon^{19/4} & \varepsilon^{19/4} \\ \varepsilon^{17/2} & \varepsilon^{5/2} & \varepsilon^{5/2} \end{bmatrix}, \tag{6.6}$$

for

$$\begin{aligned} r_d' &= l_1 - l_3 = 11\,, \qquad s = -\frac{9}{4}, \qquad s' = \frac{35}{4}\,, \qquad k_d = -\frac{5}{2}, \quad \text{or} \\ r_d' &= l_1 - l_3 = -11\,, \qquad s = \frac{9}{4}, \qquad s' = -\frac{35}{4}\,, \qquad k_d = \frac{5}{2}\,. \end{aligned} \tag{6.7}$$

Solution 2, $|s' + s| = 6$,

$$Y^u = \begin{bmatrix} \varepsilon^{16} & \varepsilon^{6} & \varepsilon^{8} \\ \varepsilon^{6} & \varepsilon^{4} & \varepsilon^{2} \\ \varepsilon^{8} & \varepsilon^{2} & 1 \end{bmatrix}, \qquad Y^d = \begin{bmatrix} \varepsilon^{31/2} & \varepsilon^{11/2} & \varepsilon^{11/2} \\ \varepsilon^{11/2} & \varepsilon^{9/2} & \varepsilon^{9/2} \\ \varepsilon^{15/2} & \varepsilon^{5/2} & \varepsilon^{5/2} \end{bmatrix}, \tag{6.8}$$

for

$$\begin{aligned} r_d' &= l_1 - l_3 = 10\,, \qquad s = -2\,, \qquad s' = 8\,, \qquad k_d = -\frac{5}{2}, \quad \text{or} \\ r_d' &= l_1 - l_3 = -10\,, \qquad s = 2\,, \qquad s' = -8\,, \qquad k_d = \frac{5}{2}\,. \end{aligned} \tag{6.9}$$

Solution 3, $|s + s'| = 11/2$,

$$Y^u = \begin{bmatrix} \varepsilon^{29/2} & \varepsilon^{11/2} & \varepsilon^{29/4} \\ \varepsilon^{11/2} & \varepsilon^{7/2} & \varepsilon^{7/4} \\ \varepsilon^{29/4} & \varepsilon^{7/4} & 1 \end{bmatrix}, \qquad Y^d = \begin{bmatrix} \varepsilon^{31/2} & \varepsilon^{21/4} & \varepsilon^{21/4} \\ \varepsilon^{21/4} & \varepsilon^{15/4} & \varepsilon^{15/4} \\ \varepsilon^{33/4} & \varepsilon^{2} & \varepsilon^{2} \end{bmatrix}, \tag{6.10}$$

for

$$\begin{aligned} r_d' &= l_1 - l_3 = \frac{41}{4}\,, \qquad s = -\frac{7}{4}, \qquad s' = \frac{29}{4}\,, \qquad k_d = -2\,, \quad \text{or} \\ r_d' &= l_1 - l_3 = -\frac{41}{4}, \qquad s = \frac{7}{4}, \qquad s' = -\frac{29}{4}, \qquad k_d = 2\,. \end{aligned} \tag{6.11}$$

All the previous solutions eqs. (6.7)–eqs. (6.11) lead to small $\tan\beta$ ($O(1)$), due to the choice of k_d. To find solutions such that $\tan\beta$ is $O(10)$ is more difficult, due to the requirements in the up sector, but we have found the following solution

$$\begin{aligned} r_d' &= l_1 - l_3 = \frac{19}{2}\,, \qquad s = -2\,, \qquad s' = \frac{15}{2}\,, \qquad k_d = -\frac{3}{2}, \quad \text{or} \\ r_d' &= l_1 - l_3 = \frac{-19}{2}\,, \qquad s = 2\,, \qquad s' = -\frac{15}{2}, \qquad k_d = \frac{3}{2}\,. \end{aligned} \tag{6.12}$$

6.2 The neutrino sector

Now we construct solutions for the lepton sector constrained by the requirements from the quark sector in the previous subsection, where we assumed $r_d = l_2 - l_3 = 0$, and determined the charge differences $r'_d = l_1 - l_2$ that agree with the GST relation. Indeed it is convenient to label the solutions in the previous subsection by the value of $r'_d = l_1 - l_2$. Here we find the charges n_i, l_i, and σ which satisfy the conditions arising from the neutrino sector, eqs. (5.7)–(5.9).[6] In order to satisfy eq. (5.7), the most natural solution to achieve t^ν_{12} large is to have

$$|l'_1 + n_2| = |l'_2 + n_2| \,. \tag{6.13}$$

The simplest solution is to assume that $n_2 = 0$. Since l'_1 and l'_2 are related through $r'_d = l_1 - l_3 = l'_1 - l'_2$ the solutions to this equation are:

$$r'_d = 0 \tag{6.14}$$

$$l'_1 = \frac{r'_d}{2} = -l'_2 \,. \tag{6.15}$$

Since none of the solutions found in the previous subsection had $r'_d = 0$, we have to work with the second solution in eq. (6.15). However, we do not need to solve eq. (5.9) exactly, so we are going to perturb away from it, by keeping $n_2 \neq 0$, but we expect it to be small in comparison with $l'_1 = -l'_2$. Then we write:

$$p_{12} = |l'_1 + n_2| - |l'_2 + n_2| \,. \tag{6.16}$$

So t^ν_{12} is $O(\varepsilon^{p_{12}})$. The solution eq. (6.15) implies that l'_1 and l'_2 should have opposite sign, so we choose the case $l'_1 > 0$ (the other case is similar). Since r'_d is large for all three GST solutions, and n_2 should be small in order to satisfy eq. (6.16), we can see that $|l'_2 + n_2| = -(l'_2 + n_2)$, and $|l'_1 + n_2| = l'_1 + n_2$ for all the solutions from the previous subsection. Putting these relations into eq. (6.16) we get:

$$n_2 = \frac{p_{12}}{2} \,. \tag{6.17}$$

So when we choose p_{12}, n_2 is determined. Now for the t^ν_{13} mixing, which should be at most $O(\lambda)$, from eq. (5.8) we need

$$|l'_1 + n_3| > |l'_2 + n_3| \Rightarrow n_3 > 0 \,, \tag{6.18}$$

hence let us define p_{13} by:

$$p_{13} = |l'_1 + n_3| - |l'_2 + n_3| \,, \tag{6.19}$$

We assume that the first term in eq. (5.8) dominates. Then $t^\nu_{13} \approx \varepsilon^{p_{13}}/\sqrt{2}$.[7] By applying the same logic that led to eq. (6.17), we achieve:

$$n_3 = \frac{p_{13}}{2} \,. \tag{6.20}$$

[6] The condition $l_2 = l_3$ is a requirement of the class of see-saw models that we are looking for, single right-handed neutrino dominance (SRHND). Note that here we can also have $l'_2 = -l'_3$ which then forces $n_3 = 0$ for $l'_2 \neq 0$, in which case the solutions will be even more restricted.

[7] We have checked that this is indeed true for the solutions that we find for n_2, n_3 later in this section.

So fixing $p_{13} \geq 1$ we fix n_3. Now we need to impose the conditions under which we can have an appropriate value of eq. (5.12). First note that in order to achieve $m_{\nu_3} = O(10^{-2})$eV:

$$\begin{aligned} \text{for}\langle\Sigma\rangle = M_P\,, \qquad & \frac{v^2}{\langle\Sigma\rangle} \approx 6\times 10^{-6}\,\text{eV} \quad \text{we need } \varepsilon^{p_3} \sim 10^4 \\ \text{for}\langle\Sigma\rangle = M_G\,, \qquad & \frac{v^2}{\langle\Sigma\rangle} \approx 6\times 10^{-3}\,\text{eV} \quad \text{we need } \varepsilon^{p_3} \sim 10\,, \end{aligned} \tag{6.21}$$

where p_3 has been defined in eq. (5.13). In terms of powers of λ, we have $\lambda^{-4} - \lambda^{-7} = O(10^5) - O(10^4)$ for $\langle\Sigma\rangle = M_P$ and $\lambda^{-1}, \lambda^{-2} = O(10)$ for $\langle\Sigma\rangle = M_G$. This corresponds to the following requirements:

$$\text{for}\langle\Sigma\rangle = M_P\,, \qquad p_3 = (-4, -7) \tag{6.22}$$

$$\text{for}\langle\Sigma\rangle = M_G\,, \qquad p_3 = (-1, -2)\,. \tag{6.23}$$

We can conclude that for zero n_2, from eq. (5.4), since $n_3 > 0$, so must σ be positive. Then we can write the power $p_2 - p_3$ ($m_{\nu_2}/m_{\nu_3} \sim \varepsilon^{p_2 - p_3}$) as follows:

$$p_2 - p_3 = -2(l'_2 + n_2) - (2n_2 + \sigma) + (2n_3 + \sigma) \mp 2(l'_2 + n_3)\,. \tag{6.24}$$

The uncertainty in the final sign comes from whether $|l'_2| > |n_3|$. If this is the case then we get:

$$p_3 - p_2 = 4(n_2 - n_3)\,. \tag{6.25}$$

Otherwise we end up with

$$p_2 - p_3 = -4(l'_2 + n_2)\,. \tag{6.26}$$

The second form is of no use to us, since we know that $-l'_2$ is big for the models we are considering, and since n_2 is small we can not get an acceptable mass ratio for m_{ν_2} to m_{ν_3}. For the first form, eq. (6.25), we need $n_2 \neq 0$, because substituting eq. (6.20) into eq. (6.25) we have $p_2 - p_3 = 2p_{13} - 4n_2$ and we need $p_{13} \geq 1$ so for $n_2 = 0$ we have $p_2 - p_3 \geq 2$.

With the above requirements then we can see that the parameters n_3 and n_2 do not depend on r'_d. The only parameter which depends on this is σ, through eq. (6.24), using the fact that $l'_2 = -r'_d/2$. This also fixes the scale at which the U(1) should be broken. So, independently of r'_d, we have the following solutions

$$\begin{aligned} p_{12} &= \frac{1}{4}\,, \quad p_{13} = 1\,, \quad p_2 - p_3 = \frac{3}{2} \Rightarrow n_2 = \frac{1}{8}\,, \quad n_3 = \frac{1}{2}\,; \\ p_{12} &= \frac{1}{2}\,, \quad p_{13} = 1\,, \quad p_2 - p_3 = 1 \Rightarrow n_2 = \frac{1}{4}\,, \quad n_3 = \frac{1}{2}\,. \end{aligned} \tag{6.27}$$

We can write the approximate expressions of mixings and masses in terms of the above results and the coefficients a^ν_{ij} of $O(1)$,

$$\begin{aligned} t^\nu_{23} &= \frac{a^\nu_{23}}{a^\nu_{33}}\,, \quad t^\nu_{13} = \frac{a^\nu_{13}\varepsilon^{|2n_3|}}{\sqrt{{a^\nu_{33}}^2 + {a^\nu_{23}}^2}}\,, \quad t^\nu_{12} = \frac{a^\nu_{12}\varepsilon^{|2n_2|}}{(c_{23}a^\nu_{22} - s_{23}a^\nu_{32})}\,, \\ \frac{m_{\nu_2}}{m_{\nu_3}} &= \frac{{c^\nu_{23}}^2}{{c^\nu_{12}}^2}\frac{(a^\nu_{22} - a^\nu_{32}t_{23})^2}{({a^\nu_{33}}^2 + {a^\nu_{23}}^2)}\varepsilon^{|4(n_3 - n_2)|}\,, \quad m_{\nu_3} = \frac{v^2}{\langle\Sigma\rangle}({a^\nu_{33}}^2 + a^{\nu 2}_{23})\varepsilon^{|p_3|}\,. \end{aligned} \tag{6.28}$$

Sol.	r'_d	n_2	n_3	p_3	σ	M_3 [GeV]
1	11	$\frac{1}{8}$	$\frac{1}{2}$	$(-4,-7)$	(14,16)	$O(10^{10}), O(10^8)$
1	11	$\frac{1}{4}$	$\frac{1}{2}$	$(-4,-7)$	(14,16)	$O(10^{10}), O(10^8)$
2	10	$\frac{1}{8}$	$\frac{1}{2}$	$(-4,-7)$	(13,14)	$O(10^{11}), O(10^9)$
2	10	$\frac{1}{4}$	$\frac{1}{2}$	$(-4,-7)$	(13,14)	$O(10^{11}), O(10^9)$
3	$\frac{41}{4}$	$\frac{1}{8}$	$\frac{1}{2}$	$(-4,-7)$	$(\frac{53}{4}, \frac{57}{4})$	$O(10^{10}),O(10^8)$
3	$\frac{41}{4}$	$\frac{1}{4}$	$\frac{1}{2}$	$(-4,-7)$	$(\frac{53}{4}, \frac{57}{4})$	$O(10^{10}),O(10^8)$

Table 3: Σ at M_P for the solutions satisfying the GST relation.

Sol.	r'_d	n_2	n_3	p_3	σ	M_3 [GeV]
1	11	$\frac{1}{8}$	$\frac{1}{2}$	$(-1,-2)$	(10, 11)	$O(10^8)$
1	11	$\frac{1}{4}$	$\frac{1}{2}$	$(-1,-2)$	(10, 11)	$O(10^8)$
2	10	$\frac{1}{8}$	$\frac{1}{2}$	$(-1,-2)$	(10, 11)	$O(10^9), O(10^{10})$
2	10	$\frac{1}{4}$	$\frac{1}{2}$	$(-1,-2)$	(10, 11)	$O(10^9), O(10^{10})$
3	$\frac{41}{4}$	$\frac{1}{8}$	$\frac{1}{2}$	$(-1,-2)$	$(\frac{37}{4}, \frac{41}{4})$	$O(10^9), O(10^8)$
3	$\frac{41}{4}$	$\frac{1}{4}$	$\frac{1}{2}$	$(-1,-2)$	$(\frac{37}{4}, \frac{41}{4})$	$O(10^9), O(10^8)$

Table 4: Σ at M_G for the solutions satisfying the GST relation.

As we have seen above, the charges σ are constrained by the differences r'_d, the requirements of eq. (6.24) and the solutions to eq. (6.27), which have the same value for n_3, so for these two sets of solutions we have the same value for σ. We write down these solutions for $\langle\Sigma\rangle = M_P$ in table 3 and for $\langle\Sigma\rangle = M_G$ in table 4.

The solutions presented here satisfy the conditions of the single neutrino right-handed dominance, eq. (5.4), which relate second and third families. For the first and second family we need similar conditions, which are safely satisfied whenever $2n_1 > 2n_2 > -\sigma$ for $(2n_i + \sigma)$ positive. Thus n_1 is not completely determined but we can choose it to be a negative number between $-\sigma/2$ and 0.

Now that we have determined the conditions that the charges l'_i and n_i need to satisfy in order to produce SRHND solutions we can determine the e_i and l_i charges, which are in agreement with the cancellation of anomalies, eqs. (2.5)–(2.8), and that determines the matrices Y^e, Y^u and Y^d. In section 4 we presented the conditions that the fermion mass matrices Y^u, Y^d, Y^e and Y^ν need to satisfy in order to give acceptable predictions for mass ratios and mixings but without specifying the charges. The charges are then determined from r'_d and k_d. We start by rewriting k_d using the SU(5) charge relations, and the fact that $l'_i \equiv l_i + h_u$:

$$k_d = q_3 + d_3 + h_d = e_3 + l_3 - h_u = e_3 + (l'_3 - h_u) - h_u\,. \tag{6.29}$$

Then we use the fact that $k_u = 0 = 2e_3 + h_u$, and we can solve for e_3 in terms of k_d and r'_d (using eq. (6.15):

$$e_3 = \frac{2k_d + r'_d}{10}\,. \tag{6.30}$$

Sol.	r'_d	k_d	n_2	n_3	e_1	e_2	e_3	l_1	l_3	Fit
1	11	$\frac{-5}{2}$	$\frac{1}{8}$	$\frac{1}{2}$	$\frac{187}{20}$	$\frac{-33}{20}$	$\frac{3}{5}$	$\frac{67}{10}$	$\frac{-43}{10}$	-
1	11	$\frac{-5}{2}$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{187}{20}$	$\frac{-33}{20}$	$\frac{3}{5}$	$\frac{67}{10}$	$\frac{-43}{10}$	-
2	10	$\frac{-5}{2}$	$\frac{1}{8}$	$\frac{1}{2}$	8	-2	0	$\frac{15}{2}$	$\frac{-5}{2}$	-
2	10	$\frac{-5}{2}$	$\frac{1}{4}$	$\frac{1}{2}$	8	-2	0	$\frac{15}{2}$	$\frac{-5}{2}$	1
3	$\frac{41}{4}$	-2	$\frac{1}{8}$	$\frac{1}{2}$	$\frac{63}{8}$	$\frac{9}{8}$	$\frac{25}{8}$	$\frac{51}{8}$	$\frac{-31}{8}$	-
3	$\frac{41}{4}$	-2	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{63}{8}$	$\frac{9}{8}$	$\frac{25}{8}$	$\frac{51}{8}$	$\frac{-31}{8}$	-

Table 5: Charged lepton charges for the SU(5) type solutions with $u = v = 0$ satisfying the GST relation. The fits are discussed in section 9.

Once we have e_3, and l'_3, we can get l_3 since $h_u = -2e_3$. From there, we can calculate the other charges from s, s', r, r' using eq. (3.3) and eq. (3.4).

The charges calculated in this way are laid out in table 5.

6.3 Solutions for the extended SU(5) case with $u + v \neq 0$

For this class of solutions, it is clear from eq. (2.35) and eq. (3.2) that the quark sector results will be unchanged. This happens since s, s', r, r' are blind to whether the family charges are related by the SU(5) relation, or the extended SU(5) relation. k_u must always be zero, and the parameterization happens to leave k_d unchanged. Since k_e is not unchanged, as discussed in section 2.4.1, we need to find k_e in order to know the structure of the electron Yukawa matrix.

It is helpful to rewrite k_e and k_d, from the form in eq. (2.40) by using eqs. (2.12), (2.36) and $k_u = 0$:

$$\begin{aligned} k_d &= l_3 + 3e_3 + u + \frac{4}{3}m\,, \\ k_e &= l_3 + 3e_3 + u + m\,, \end{aligned} \tag{6.31}$$

where we have written $u + v = m$, as we will discuss in section 10.3 m can be determined such a that the effects of the breaking of U(1) in the μ term are of order $\leq m_{3/2}$. But on the other hand we need to keep the observed relation at low energies $m_b = O(m_\tau)$, so either m has to remain small or be negative to achieve $|k_d| = O(|k_e|)$. In the present case the Y^d matrix has exactly the same form as in eqs. (2.38) and Y^e has the form

$$Y^e = \begin{pmatrix} \varepsilon^{|s'+r'_d+k_e|} & \varepsilon^{|s+r'_d+k_e|} & \varepsilon^{|r'_d+k_e|} \\ \varepsilon^{|s'+r_d+k_e|} & \varepsilon^{|s+r_d+k_e|} & \varepsilon^{|r_d+k_e|} \\ \varepsilon^{|s'+k_e|} & \varepsilon^{|s+k_e|} & \varepsilon^{|k_e|} \end{pmatrix}. \tag{6.32}$$

With $l_2 = l_3$, which determines the solutions of the charges e_i and l_i compatible with the condition $r_d = l_2 - l_3 = 0$, the discussion follows exactly as section 6.2 because there we have not referred to other parameters than to k_d, r, r', s and s' without specifying their relations with the charges cancelling the anomalies.

Sol.	r'_d	k_d	u	v	k_e	e_1	e_2	e_3	l_1	l_3	n_2	n_3	Fit
1	11	$\frac{-5}{2}$	$-\frac{13}{2}$	7	$\frac{8}{3}$	$\frac{709}{60}$	$\frac{49}{60}$	$\frac{46}{15}$	$\frac{77}{15}$	$\frac{-88}{15}$	$\frac{1}{8}$	$\frac{1}{2}$	-
1	11	$\frac{-5}{2}$	$\frac{-13}{2}$	7	$\frac{8}{3}$	$\frac{709}{60}$	$\frac{49}{60}$	$\frac{46}{15}$	$\frac{77}{15}$	$\frac{-88}{15}$	$\frac{1}{4}$	$\frac{1}{2}$	-
2	10	$\frac{-5}{2}$	-13	$\frac{27}{2}$	$\frac{8}{3}$	$\frac{407}{30}$	$\frac{107}{30}$	$\frac{167}{30}$	$\frac{47}{15}$	$\frac{-103}{15}$	$\frac{1}{8}$	$\frac{1}{2}$	2
2	10	$\frac{-5}{2}$	-13	$\frac{27}{2}$	$\frac{8}{3}$	$\frac{407}{30}$	$\frac{107}{30}$	$\frac{167}{30}$	$\frac{47}{15}$	$\frac{-103}{15}$	$\frac{1}{4}$	$\frac{1}{2}$	-
3	$\frac{41}{4}$	-2	$\frac{-10}{3}$	$\frac{23}{6}$	$\frac{7}{3}$	$\frac{363}{40}$	$\frac{3}{40}$	$\frac{73}{40}$	$\frac{653}{120}$	$\frac{-577}{120}$	$\frac{1}{8}$	$\frac{1}{2}$	3
3	$\frac{41}{4}$	-2	$\frac{-10}{3}$	$\frac{23}{6}$	$\frac{7}{3}$	$\frac{363}{40}$	$\frac{3}{40}$	$\frac{73}{40}$	$\frac{653}{120}$	$\frac{-577}{120}$	$\frac{1}{4}$	$\frac{1}{2}$	-

Table 6: Charged lepton charges for the extended SU(5) solutions with $m = u+v = 1/2$ satisfying the GST relation. The fits are discussed in section 9.

In this case, the analysis that leads to eq. (6.30) must be repeated, but accounting for the fact that instead of the SU(5) relation between the charges, we must instead use the extended SU(5) relation between the charges. In this case, we find that:

$$k_d = 3e_3 + l_3 + u + \frac{4}{3}(u+v) - 2h_u = 5e_3 + l'_3 + \frac{10}{3}u + \frac{4}{3}v\,, \tag{6.33}$$

where we have used that $l'_i = l_i - 2e_3 - u$. l'_i is defined in such a way that $l_i + n_j + h_u = l'_i + n_j$. Using again the fact that $l'_3 = l'_2 = -\frac{r'_d}{2}$, we find that:

$$e_3 = \frac{1}{10}\left(2k_d + r'_d - \frac{20}{3}u - \frac{8}{3}v\right). \tag{6.34}$$

Using these results, and the values of s, s', r_f, r'_f, we can find the charges in table 6.

7. SU(5) solutions not satisfying the GST relation

7.1 The quark sector

As we can see the GST relation puts a constraint on the opposite signs of s and s' and on the difference of $r'_d = l_1 - l_3$. If we do not impose these requirements, allowing all the numbers s, s', r, r' and k_d to have the same sign, positive or negative, we can factorize the k_d factor out of the Y^d matrix and so can write the down matrix in the form

$$Y^d = \varepsilon^{|k_d|}\begin{bmatrix} \varepsilon^{|s'+l_1-l_3|} & \varepsilon^{|s'|} & \varepsilon^{|s'|} \\ \varepsilon^{|s+l_1-l_3|} & \varepsilon^{|s|} & \varepsilon^{|s|} \\ \varepsilon^{|l_1-l_3|} & 1 & 1 \end{bmatrix}. \tag{7.1}$$

In this case we do not have the restriction $|s+l_1-l_3| = |s'|$ so the parameter $l_1 - l_2$ is not fixed by these conditions. In these cases k_d is not constrained so it can acquire a value in the range $\sim (0,3)$ for different values of $\tan\beta$. In these cases all positive or all negative charges, the cases which reproduce quark masses and mixings are for

$$|s| = 2\,, \qquad |s'| = 3 \qquad \text{or} \qquad |s| = 2\,, \qquad |s'| = 4\,. \tag{7.2}$$

For $|s| = 2,\ |s'| = 3$ we have

$$Y^d = \varepsilon^{|k_d|}\begin{bmatrix} \varepsilon^{|3+l_1-l_3|} & \varepsilon^{|3|} & \varepsilon^{|3|} \\ \varepsilon^{|2+l_1-l_3|} & \varepsilon^{|2|} & \varepsilon^{|2|} \\ \varepsilon^{|l_1-l_3|} & 1 & 1 \end{bmatrix}. \tag{7.3}$$

For $|s|=2$, $|s'|=4$ we have

$$Y^d = \varepsilon^{|k_d|}\begin{bmatrix} \varepsilon^{|4+l_1-l_3|} & \varepsilon^{|4|} & \varepsilon^{|4|} \\ \varepsilon^{|2+l_1-l_3|} & \varepsilon^{|2|} & \varepsilon^{|2|} \\ \varepsilon^{|l_1-l_3|} & 1 & 1 \end{bmatrix}. \tag{7.4}$$

From eq. (7.3) and eq. (7.4) we can check if certain differences of leptonic charges can yield a suitable quark phenomenology. From eq. (4.2) we can see that in the cases of having all charges l_i and e_i either positive or negative, then all the terms contributing to the first eigenvalue of Y^d, y_1, will have the same power, as we mentioned earlier. So the difference r'_d here is constrained to reproduce an appropriate ratio m_d/m_s. Let us take here for definitiveness the case for positive charges (the negative charges case is completely analogous). Thus for $s=2$, $s'=3$, we have

$$\frac{m_d}{m_s} \sim \frac{\varepsilon^{3+r'_d}}{\varepsilon^2} \sim (\lambda^2, \lambda^{3/2}) \tag{7.5}$$

so in this case we have $r'_d = 1,\ 3/2$. For the case $s=2$, $s'=4$, we have

$$\frac{m_d}{m_s} \sim \frac{\varepsilon^{4+r'_d}}{\varepsilon^2} \sim (\lambda^2, \lambda^{3/2}) \tag{7.6}$$

we do not want $r'_d = 0$ as it will give somewhat large contribution from the $(3,1)$ element of the Y^d matrix to the eigenvalues. So for this case $r'_d \approx 1/2$. In this case we have the following matrices for eq. (7.3)

$$Y^d = \begin{bmatrix} \epsilon^4 & \epsilon^3 & \epsilon^3 \\ \epsilon^3 & \epsilon^2 & \epsilon^2 \\ \epsilon^1 & 1 & 1 \end{bmatrix} \varepsilon^{k_d}, \qquad Y^d = \begin{bmatrix} \epsilon^{9/2} & \epsilon^4 & \epsilon^4 \\ \epsilon^{7/2} & \epsilon^2 & \epsilon^2 \\ \epsilon^{3/2} & 1 & 1 \end{bmatrix} \varepsilon^{k_d}, \tag{7.7}$$

respectively for $r'_d = 1,\ 3/2$. For eq. (7.4) we have

$$Y^d = \begin{bmatrix} \epsilon^{9/2} & \epsilon^4 & \epsilon^4 \\ \epsilon^{5/2} & \epsilon^2 & \epsilon^2 \\ \epsilon^{1/2} & 1 & 1 \end{bmatrix} \epsilon^{k_d}, \tag{7.8}$$

for $r'_d = 1/2$. These solutions work for $k_d \in (0,3)$, depending on the value of $\tan\beta$, these matrices yield acceptable phenomenology in both charged the lepton sector and d quark sector.

7.2 The neutrino sector

As we have seen in section 7.1, in these cases r'_d is constrained to be $r'_d \in (1, 3/2)$ for $(s,s') = (2,3)$ and $r'_d \approx 1/2$ for $(s,s') = (2,4)$ but let us leave it unspecified for the moment. We consider here the case of all the parameters related to l_i and e_i positive. In this case we require that all the neutrino charges, n_i to be negative but σ positive. We proceed as in section 6.1, in order to identify the charges l'_i, n_i and σ. In principle we need $\varepsilon^{|l'_1+n_2|} = \varepsilon^{|l'_2+n_2|}$ but now we require $l'_1, l'_2 \geq 0$ so now the appropriate solution to this would be

$$l_1' = r_d', \qquad l_2' = 0, \qquad n_2 = \frac{-r_d'}{2}. \tag{7.9}$$

However in this case, as in the case of section 6.1, we will only be able to produce $m_{\nu_2}/m_{\nu_3} \sim \varepsilon^2$. So we work with a solution of the form eq. (6.16). For this case we then have

$$l_1' = r_d', \qquad l_2' = 0, \qquad n_2 = \frac{p_{12} - r_d'}{2}. \tag{7.10}$$

Note that in this case the charges l_i are positive because $l_2 = k_d - 3e_3$ and here $e_3 = k_d$. For t_{13} we also make use of the parameterization of eq. (6.19). Assuming that $|r_d'| > |n_3|$,

$$n_3 = \frac{p_{13} - r_d'}{2}. \tag{7.11}$$

In order to achieve an appropriate ratio for m_{ν_2}/m_{ν_3} we need now the conditions $2n_3 + \sigma > 0$, $2n_2 + \sigma > 0$, $l_2' + n_2 < 0$, $l_2' + n_3 < 0$, for one of the last two inequalities the equality can be satisfied, but not for both. For this case, we have also $p_2 - p_3 = 4(n_3 - n_2)$ and using eq. (7.10) and eq. (7.11) we have $p_2 - p_3 = 2(p_{13} - p_{12})$. We can also choose the parameters p_{12}, p_{13} and $p_2 - p_3$ as in eq. (6.27) but now n_3 and n_2 are given by eq. (7.10) and eq. (7.11). Thus we have

$$\begin{aligned}
&p_{12} = \frac{1}{4}, \qquad p_{13} = 1, \qquad p_2 - p_3 = \frac{3}{2} \\
&\rightarrow\ n_2 = \frac{1}{8} - \frac{r_d'}{2} < 0, \qquad n_3 = \frac{1}{2} - \frac{r_d'}{2} < 0 \Rightarrow r_d' \geq 1, \qquad (7.12) \\
&p_{12} = \frac{1}{2}, \qquad p_{13} = 1, \qquad p_2 - p_3 = 1 \\
&\rightarrow\ n_2 = \frac{1}{4} - \frac{r_d'}{2} < 0, \qquad n_3 = \frac{1}{2} - \frac{r_d'}{2} < 0 \Rightarrow r_d' \geq 1. \qquad (7.13)
\end{aligned}$$

In section 4 we determined the approximate values for r_d'. For $(s, s') = (2, 3)$ we can have $r_d' = 1, 3/2$ while for $(s, s') = (2, 4)$ we have $r_d' \approx 1/2$, which however is not in agreement with the conditions of eq. (7.12) and eq. (7.13). The approximate expressions of mixings and masses in terms of the above results and the coefficients a^ν_{ij} of $O(1)$ are as in eq. (6.28), except for t^ν_{13} and t^ν_{12} which now read

$$t^\nu_{13} = \frac{a^\nu_{13}\varepsilon^{|r_d'+n_3|-|n_3|}}{\sqrt{{a^\nu_{33}}^2 + {a^\nu_{23}}^2}}, \qquad t^\nu_{12} = \frac{a^\nu_{12}\varepsilon^{(|r_d'+n_2|-|n_2|)}}{(c_{23}a^\nu_{22} - s_{23}a^\nu_{32})}. \tag{7.14}$$

We have listed the possible solutions of table 7 for eq. (7.12) at $\langle\Sigma\rangle = M_P$ and in table 8 for $\langle\Sigma\rangle = M_G$.

7.3 Solutions for the extended SU(5) case with $u + v \neq 0$

We do not present any charges for this class of solutions, but here is how the charges would be calculated. In this case, the analysis is carried out in the same way as in section 6.3. The only subtlety is that the relation linking l_2' to r_d' is different. Instead, we have, from eq. (7.10) that $l_2' = l_3' = 0$. Putting this result into eq. (6.33), we achieve:

$$e_3 = \frac{1}{10}\left(2k_d + \frac{4}{3}u - \frac{8}{3}v\right). \tag{7.15}$$

r'_d	n_2	n_3	p_3	σ	$M_3[GeV]$
1	$\frac{-3}{8}$	0	$(-\frac{9}{2},-\frac{5}{2})$	$(\frac{9}{2},5)$	$O(10^{15})$
$\frac{3}{2}$	$\frac{-5}{8}$	$\frac{-1}{4}$	$(-\frac{17}{4},-\frac{19}{4})$	$(5,\frac{11}{2})$	$O(10^{15})$
1	$\frac{-1}{4}$	1	(-6,-7)	(6,7)	$O(10^{15})$, $O(10^{14})$
$\frac{3}{2}$	$\frac{-1}{2}$	$\frac{-1}{4}$	(-6,-7)	$(\frac{27}{4},\frac{31}{4})$	$O(10^{14})$, $O(10^{15})$

Table 7: Σ at M_P for the solutions not satisfying the GST relation.

r'_d	n_2	n_3	p_3	σ	M_3 [GeV]
1	$\frac{-3}{8}$	0	$(0,-\frac{1}{2})$	$(0,\frac{1}{2})$	$O(10^{15})$
$\frac{3}{2}$	$\frac{-5}{8}$	$\frac{-1}{4}$	$(-\frac{1}{4},-\frac{3}{4})$	$(1,\frac{3}{2})$	$O(10^{15})$,
1	$\frac{-1}{4}$	0	1	(-1,-2)	$O(10^{18})$
$\frac{3}{2}$	$\frac{-1}{2}$	$\frac{-1}{4}$	(-1,-2)	$(\frac{7}{4},\frac{11}{4})$	$O(10^{18})$,$O(10^{17})$

Table 8: Σ at M_G for the solutions not satisfying the GST relation.

r'_d	k_d	e_1	e_2	e_3	l_1	l_3	Fit
1	2	$\frac{17}{5}$	$\frac{12}{5}$	$\frac{2}{5}$	$\frac{9}{5}$	$\frac{4}{5}$	4
$\frac{3}{2}$	2	$\frac{17}{5}$	$\frac{12}{5}$	$\frac{2}{5}$	$\frac{7}{10}$	$\frac{4}{5}$	5
1	3	$\frac{18}{5}$	$\frac{13}{5}$	$\frac{3}{5}$	$\frac{7}{10}$	$\frac{-3}{10}$	-
$\frac{3}{2}$	3	$\frac{18}{5}$	$\frac{13}{5}$	$\frac{3}{5}$	$\frac{6}{5}$	$\frac{-3}{10}$	-

Table 9: Charged lepton $U(1)_X$ charges for the solutions $u = v = 0$ not satisfying the GST relation. The fits are discussed in section 9.

From e_3 and l'_3, the other charges may be calculated using the known values of s, s', r_d, r'_d, u and v, by using the extended SU(5) charge relations, eq. (2.36) and the simplified parametrization, eq. (3.5).

8. The non-SU(5) cases

8.1 Solutions for $u = v = 0$, in the Pati-Salam case

With $l_2 = l_3$, in this case we have $s = q_2 - q_3 = l_2 - l_3 = 0$ then the charges of the matrices are

$$C(Y^{u,\nu}) = \begin{bmatrix} l_1 - l_3 + e_1 - e_3 & l_1 - l_3 + e_2 - e_3 & l_1 - l_3 \\ e_1 - e_3 & e_2 - e_3 & 0 \\ e_1 - e_3 & e_2 - e_3 & 0 \end{bmatrix}$$

$$C(Y^{d,l}) = \begin{bmatrix} l_1 + l_3 + e_1 + e_3 & l_1 + l_3 + e_2 + e_3 & l_1 + l_3 + 2e_3 \\ 2l_3 + e_1 + e_3 & 2l_3 + e_2 + e_3 & 2l_3 + 2e_3 \\ 2l_3 + e_1 + e_3 & 2l_3 + e_2 + e_3 & 2l_3 + 2e_3 \end{bmatrix} \quad (8.1)$$

In this case the $U(1)_X$ symmetry does not give an appropriate description of fermion masses and mixings, however it can be combined with non-renormalizable operators of the Pati-Salam group, [30], in order to give a good description of the fermion phenomenology.

8.2 Solutions for $u + v = 0$

One trivial example of non- SU(5) cases was given in section 2.3.1 for the solution $u+v = 0$. We proceed as in the section 4 — in order to analyze the appropriate phenomenology. We are interested in the cases $l_2 = l_3$, this together with the condition of $O(1)$ top Yukawa coupling give us the following matrices of charges, which are derived with the appropriate substitutions in eq. (2.31)–eq. (2.33),

$$\begin{aligned}
C(Y^d) &= \begin{bmatrix} l_1 + e_1 & \frac{4(l_3-l_1)}{3} + e_1 - e_3 & e_2 - e_3 \\ \frac{l_3-l_1}{3} + e_2 - e_3 & e_2 - e_3 & e_2 - e_3 \\ \frac{l_3-l_1}{3} & 0 & 0 \end{bmatrix}, \\
C(Y^u) &= \begin{bmatrix} l_1 + e_1 & \frac{2(l_3-l_1)}{3} + e_3 - e_1 & \frac{2(l_3-l_1)}{3} + e_3 - e_1 \\ \frac{l_3-l_1}{3} + e_3 - e_3 & e_3 - e_2 & e_3 - e_2 \\ \frac{l_3-l_1}{3} & 0 & 0 \end{bmatrix}, \\
C(Y^e) &= \begin{bmatrix} l_1 + e_1 & l_1 + e_2 & l_1 + e_3 \\ e_2 - e_3 & e_2 - e_3 & 0 \\ e_2 - e_3 & e_2 - e_3 & 0 \end{bmatrix}.
\end{aligned} \tag{8.2}$$

Due to the form of the charges in the up and down quark matrices, first at all we would need two expansion parameters: ϵ^u and ϵ^d. But with this structure alone it is not possible to account simultaneously for appropriate mass ratios of the second to third family of quarks and for an appropriate V_{cb} mixing. So in this case just with a U(1) it is not possible to explain fermion masses and mixings in the context of the single neutrino right-handed dominance, SNRHD.

9. Numerical fits of masses and mixings

9.1 Fitted examples

In this section we present numerical fits to some of the examples detailed in sections 6,7 and we compare the results with a fit of a generic SU(3)-like case [6]. The simplest way to construct the lepton Yukawa matrices from the charges is to first calculate $h_{u,d}$. We extract h_d from k_e, l_3 and e_3 from $k_e = l_3 + e_3 + h_d$. In general, we can use Eq. (6.31) to obtain:

$$h_u + h_d = m = 3(k_d - k_e)\,. \tag{9.1}$$

This is then enough to construct the lepton Yukawas from the appropriate line of the table 5 or 6 of the lepton and Yukawa family charges. Below we specify the examples that we have chosen to fit.

Fit 1: SU(5) **type solution** ($u = v = 0$)**: example satisfying the GST relation.** This takes GST solution 2, (eq. (6.8)) in the SU(5) type case, with $u = v = 0$. The charges l_i, e_i, and $n_{2,3}$ are read off from the fourth line of Table 5. In principle, the value of σ

would be read off from either Table 4 (for neutrino masses generated at the GUT scale) or table 3 (for neutrino masses geneated at the Planck scale). However, these tables allow for a range of σ; for this fit, we take $\sigma = 21/2$ for GUT scale neturino mass generation, and $\sigma = 29/2$ for Planck scale neutrino mass generation.

Then, up to σ and $n_1 : -\sigma/2 \le n_1 \le 0$, the Yukawa and Majorana matrices are:

$$Y^u = \begin{bmatrix} a^u_{11}\epsilon^{16} & a^u_{12}\epsilon^{6} & a^u_{13}\epsilon^{8} \\ a^u_{21}\epsilon^{6} & a^u_{22}\epsilon^{4} & a^u_{23}\epsilon^{2} \\ a^u_{31}\epsilon^{8} & a^u_{32}\epsilon^{2} & a^u_{33} \end{bmatrix}, \qquad Y^d = \begin{bmatrix} a^d_{11}\epsilon^{31/2} & a^d_{12}\epsilon^{11/2} & a^d_{13}\epsilon^{11/2} \\ a^d_{21}\epsilon^{11/2} & a^d_{22}\epsilon^{9/2} & a^d_{23}\epsilon^{9/2} \\ a^d_{31}\epsilon^{15/2} & a^d_{32}\epsilon^{5/2} & a^d_{33}\epsilon^{5/2} \end{bmatrix}$$

$$Y^e = \begin{bmatrix} a^e_{11}\epsilon^{31/2} & a^e_{12}\epsilon^{11/2} & a^e_{13}\epsilon^{15/2} \\ a^e_{21}\epsilon^{11/2} & a^e_{22}\epsilon^{9/2} & a^e_{23}\epsilon^{5/2} \\ a^e_{31}\epsilon^{11/2} & a^e_{32}\epsilon^{9/2} & a^e_{33}\epsilon^{5/2} \end{bmatrix}, \qquad Y^\nu = \begin{bmatrix} a^\nu_{11}\epsilon^{|n_1+5/2|} & a^\nu_{12}\epsilon^{11/4} & a^\nu_{13}\epsilon^{12/4} \\ a^\nu_{21}\epsilon^{|n_1-15/2|} & a^\nu_{22}\epsilon^{29/4} & a^\nu_{23}\epsilon^{28/4} \\ a^\nu_{31}\epsilon^{|n_1-15/2|} & a^\nu_{32}\epsilon^{29/4} & a^\nu_{33}\epsilon^{28/4} \end{bmatrix}$$

$$M_{RR} = \begin{bmatrix} \epsilon^{|2n_1+\sigma|} & \epsilon^{|1/4+n_1+\sigma|} & \epsilon^{|1/2+n_1+\sigma|} \\ . & a^N_{22}\epsilon^{|1/2+\sigma|} & \epsilon^{|3/4+\sigma|} \\ . & . & \epsilon^{|1+\sigma|} \end{bmatrix} \langle\Sigma\rangle\,. \tag{9.2}$$

Fit 2: Extended SU(5) solution ($u + v \neq 0$) satisfying the GST relation. This takes GST solution 2, (eq. (6.8)), in the extended SU(5) case with $u + v \neq 0$. The charges l_i, e_i and $n_{2,3}$ are read off from the third line of table 6. The values of σ taken are $\sigma = 19/2$, $\sigma = 29/2$ for GUT scale and Planck scale neutrino mass generation respectively. Again, n_1 is taken to lie in the region $-\sigma/2 \le n_1 \le 0$.[8]

$$Y^u = \begin{bmatrix} a^u_{11}\epsilon^{16} & a^u_{12}\epsilon^{6} & a^u_{13}\epsilon^{8} \\ a^u_{21}\epsilon^{6} & a^u_{22}\epsilon^{4} & a^u_{23}\epsilon^{2} \\ a^u_{31}\epsilon^{8} & a^u_{32}\epsilon^{2} & a^u_{33} \end{bmatrix}, \qquad Y^d = \begin{bmatrix} a^d_{11}\epsilon^{31/2} & a^d_{12}\epsilon^{11/2} & a^d_{13}\epsilon^{11/2} \\ a^d_{21}\epsilon^{11/2} & a^d_{22}\epsilon^{9/2} & a^d_{23}\epsilon^{9/2} \\ a^d_{31}\epsilon^{15/2} & a^d_{32}\epsilon^{5/2} & a^d_{33}\epsilon^{5/2} \end{bmatrix}$$

$$Y^e = \begin{bmatrix} a^e_{11}\epsilon^{46/3} & a^e_{12}\epsilon^{16/3} & a^e_{13}\epsilon^{22/3} \\ a^e_{21}\epsilon^{16/3} & a^e_{22}\epsilon^{14/3} & a^e_{23}\epsilon^{8/3} \\ a^e_{31}\epsilon^{16/3} & a^e_{32}\epsilon^{14/3} & a^e_{33}\epsilon^{8/3} \end{bmatrix}, \qquad Y^\nu = \begin{bmatrix} a^\nu_{11}\epsilon^{|n_1+5|} & a^\nu_{12}\epsilon^{\frac{41}{8}} & a^\nu_{13}\epsilon^{\frac{11}{2}} \\ a^\nu_{21}\epsilon^{|n_1-5|} & a^\nu_{22}\epsilon^{\frac{39}{8}} & a^\nu_{23}\epsilon^{\frac{9}{2}} \\ a^\nu_{31}\epsilon^{|n_1-5|} & a^\nu_{32}\epsilon^{\frac{39}{8}} & a^\nu_{33}\epsilon^{\frac{9}{2}} \end{bmatrix}$$

$$M_{RR} = \begin{bmatrix} \epsilon^{|2n_1+\sigma|} & \epsilon^{|1/8+n_1+\sigma|} & \epsilon^{|1/2+n_1+\sigma|} \\ . & a^N_{22}\epsilon^{|1/4+\sigma|} & \epsilon^{|5/8+\sigma|} \\ . & . & \epsilon^{|1+\sigma|} \end{bmatrix} \langle\Sigma\rangle\,. \tag{9.3}$$

Fit 3: extended SU(5) solution ($u + v \neq 0$), satisfying the GST relation. This takes GST solution 3, (eq. (6.11)), in the extended SU(5) case with $u+v \neq 0$. The charges l_i, e_i and $n_{2,3}$ are read off from the fifth line of table 6. The values of σ taken are $\sigma = 39/4$, $\sigma = 55/4$ for GUT and Planck scale neutrino mass generation respectively. n_1 lies in the

[8] The difference between Fit 1 and Fit 2 is that the charges (tables 5 and 6 respectively) are determined in a different way and hence the value of the effective parameter expansion ε is different

region $-\sigma/2 \leq n_1 \leq 0$.

$$Y^u = \begin{bmatrix} a^u_{11}\epsilon^{38/4} & a^u_{12}\epsilon^{22/4} & a^u_{13}\epsilon^{29/4} \\ a^u_{21}\epsilon^{22/4} & a^u_{22}\epsilon^{14/4} & a^u_{23}\epsilon^{7/4} \\ a^u_{31}\epsilon^{29/4} & a^u_{32}\epsilon^{7/4} & a^u_{33} \end{bmatrix}, \qquad Y^d = \begin{bmatrix} a^d_{11}\epsilon^{62/4} & a^d_{12}\epsilon^{21/4} & a^d_{13}\epsilon^{21/4} \\ a^d_{21}\epsilon^{21/4} & a^d_{22}\epsilon^{15/4} & a^d_{23}\epsilon^{15/4} \\ a^d_{31}\epsilon^{33/4} & a^d_{32}\epsilon^{8/4} & a^d_{33}\epsilon^{8/4} \end{bmatrix}$$

$$Y^e = \begin{bmatrix} a^e_{11}\epsilon^{46/3} & a^e_{12}\epsilon^{19/3} & a^e_{13}\epsilon^{97/12} \\ a^e_{21}\epsilon^{61/12} & a^e_{22}\epsilon^{47/12} & a^e_{23}\epsilon^{13/6} \\ a^e_{31}\epsilon^{61/12} & a^e_{32}\epsilon^{47/12} & a^e_{33}\epsilon^{13/6} \end{bmatrix}, \qquad Y^\nu = \begin{bmatrix} a^\nu_{11}\epsilon^{|n_1+\frac{41}{8}|} & a^\nu_{12}\epsilon^{\frac{21}{4}} & a^\nu_{13}\epsilon^{\frac{45}{8}} \\ a^\nu_{21}\epsilon^{|n_1-\frac{41}{8}} & a^\nu_{22}\epsilon^{5} & a^\nu_{23}\epsilon^{\frac{37}{8}} \\ a^\nu_{31}\epsilon^{|n_1-\frac{41}{8}} & a^\nu_{32}\epsilon^{5} & a^\nu_{33}\epsilon^{\frac{37}{8}} \end{bmatrix}$$

$$M_{RR} = \begin{bmatrix} \epsilon^{|2n_1+\sigma|} & \epsilon^{|1/8+n_1+\sigma|} & \epsilon^{|1/2+n_1+\sigma|} \\ . & a^N_{22}\epsilon^{|1/4+\sigma|} & \epsilon^{|5/8+\sigma|} \\ . & . & \epsilon^{|1+\sigma|} \end{bmatrix} \langle \Sigma \rangle \,. \tag{9.4}$$

Fit 4: SU(5) $(u = v = 0)$ **solution not satisfying the GST relation.** Here we present a solution non satisfying the GST relation of the form of eq. (7.3) for $l_1 - l_3 = 1$, which corresponds to the set of charges of the first line of table 9. We also fix here the expansion parameter $\varepsilon = 0.19$, using the FI term. The high energy Yukawa and Majorana matrices are:

$$Y^u = \begin{bmatrix} a^u_{11}\epsilon^6 & a^u_{12}\epsilon^5 & a^u_{13}\epsilon^3 \\ a^u_{21}\epsilon^5 & a^u_{22}\epsilon^4 & a^u_{23}\epsilon^2 \\ a^u_{31}\epsilon^3 & a^u_{32}\epsilon^2 & a^u_{33} \end{bmatrix}, \qquad Y^d = \begin{bmatrix} a^d_{11}\epsilon^4 & a^d_{12}\epsilon^3 & a^d_{13}\epsilon^3 \\ a^d_{21}\epsilon^3 & a^d_{22}\epsilon^2 & a^d_{23}\epsilon^2 \\ a^d_{31}\epsilon & a^d_{32} & a^d_{33} \end{bmatrix} \epsilon^{|k_d|}$$

$$Y^e = \begin{bmatrix} a^e_{11}\epsilon^4 & a^e_{12}\epsilon^3 & a^e_{13}\epsilon \\ a^e_{21}\epsilon^3 & a^e_{22}\epsilon^2 & a^e_{23} \\ a^e_{31}\epsilon^3 & a^e_{32}\epsilon^2 & a^e_{33} \end{bmatrix} \epsilon^{|k_d|}, \qquad Y^\nu = \begin{bmatrix} a^\nu_{11}\epsilon^{|n_1+1|} & a^\nu_{12}\epsilon^{5/8} & a^\nu_{13}\epsilon \\ a^\nu_{21}\epsilon^{|n_1-3/8|} & a^\nu_{22}\epsilon^{3/8} & a^\nu_{23} \\ a^\nu_{31}\epsilon^{|n_1|} & a^\nu_{32}\epsilon^{3/8} & a^\nu_{33} \end{bmatrix},$$

$$M_{RR} = \begin{bmatrix} \epsilon^{|2n_1+\sigma|} & \epsilon^{|-3/8+n_1+\sigma|} & \epsilon^{|n_1+\sigma|} \\ . & a^N_{22}\epsilon^{|-3/4+\sigma|} & \epsilon^{|-3/8+\sigma|} \\ . & . & \epsilon^{|\sigma|} \end{bmatrix} \langle \Sigma \rangle \,. \tag{9.5}$$

Fit 5: SU(5) $(u = v = 0)$ **solution not satisfying the GST relation.** Here we present another solution non satisfying the GST relation of the form of eq. (7.3) for $l_1 - l_3 = 3/2$, which corresponds to the set of charges of the second line of table 9. We also fix here the expansion parameter $\varepsilon = 0.185$, using the FI term. The high energy Yukawa and Majorana matrices are:

$$Y^u = \begin{bmatrix} a^u_{11}\epsilon^6 & a^u_{12}\epsilon^5 & a^u_{13}\epsilon^3 \\ a^u_{21}\epsilon^5 & a^u_{22}\epsilon^4 & a^u_{23}\epsilon^2 \\ a^u_{31}\epsilon^3 & a^u_{32}\epsilon^2 & a^u_{33} \end{bmatrix}, \qquad Y^d = \begin{bmatrix} a^d_{11}\epsilon^{9/2} & a^d_{12}\epsilon^3 & a^d_{13}\epsilon^3 \\ a^d_{21}\epsilon^{7/2} & a^d_{22}\epsilon^2 & a^d_{23}\epsilon^2 \\ a^d_{31}\epsilon^{3/2} & a^d_{32} & a^d_{33} \end{bmatrix} \epsilon^{|k_d|}$$

$$Y^e = \begin{bmatrix} a^e_{11}\epsilon^{9/2} & a^e_{12}\epsilon^{7/2} & a^e_{13}\epsilon^{3/2} \\ a^e_{21}\epsilon^3 & a^e_{22}\epsilon^2 & a^e_{23} \\ a^e_{31}\epsilon^3 & a^e_{32}\epsilon^2 & a^e_{33} \end{bmatrix} \epsilon^{|k_d|}, \qquad Y^\nu = \begin{bmatrix} a^\nu_{11}\epsilon^{|n_1+1|} & a^\nu_{12}\epsilon^{5/8} & a^\nu_{13}\epsilon \\ a^\nu_{21}\epsilon^{|n_1-3/8|} & a^\nu_{22}\epsilon^{3/8} & a^\nu_{23} \\ a^\nu_{31}\epsilon^{|n_1|} & a^\nu_{32}\epsilon^{3/8} & a^\nu_{33} \end{bmatrix},$$

$$M_{RR} = \begin{bmatrix} \epsilon^{|2n_1+\sigma|} & \epsilon^{|-5/8+n_1+\sigma|} & \epsilon^{|-1/4+n_1+\sigma|} \\ . & a^N_{22}\epsilon^{|-5/4+\sigma|} & \epsilon^{|-7/8+\sigma|} \\ . & . & \epsilon^{|-1/2+\sigma|} \end{bmatrix} \langle \Sigma \rangle \,. \tag{9.6}$$

9.2 Details of the fitting method

One of the purposes of these fits is to compare which solution fits the data best while constraining the abritary coefficients to remain at $O(1)$. We therefore choose a minimization routine to find these $O(1)$ coefficients and compare the numerical values for the different solutions. In the quark sector we use eight experimental inputs in order to determine the parameters (coefficients or phases):

$$\frac{V_{ub}}{V_{cb}}, \quad \frac{V_{td}}{V_{ts}}, \quad V_{us}, \quad \mathrm{Im}\{J\}, \quad \frac{m_u}{m_c}, \quad \frac{m_c}{m_t}, \quad \frac{m_d}{m_s}, \quad \frac{m_s}{m_b}. \tag{9.7}$$

We explain in the appendix B how this fit is performed, the important point is that we can only fit eight parameters and the rest need to be fixed. The minimization algorithm has been optimized to fit the solutions satisfying the GST relation because the number of parameters is close to eight. We also fit examples of the non GST solutions but since there are more free parameters in this cases (mainly phases) it is un-practical to make a fit by fixing so many free parameters. So we present particular examples in these cases which do not necessarily correspond to the best χ^2.

In the lepton sector we perform two fits, one for the coefficients of the charged lepton mass matrix and the other for the coefficients of the neutrino mass matrix. We do not perform a combined fit for the coefficients of Y^ν and Y^e because the uncertainties in these sectors are quite different. While the uncertainties in the masses of the charged leptons is very small, the uncertainties in lepton mixings and quantities related to neutrino masses are still large, such that we cannot determine the parameters involved to a very good accuracy.

The quantities used for the fit of the coefficients of the charged lepton mass matrix are

$$\frac{m_e}{m_\mu}, \qquad \frac{m_\mu}{m_\tau}, \tag{9.8}$$

such that we can just determine two parameters, a^e_{12} and a^e_{22}, but for the cases presented here this is enough. In order to do the fit for the coefficients of the neutrino mass matrix we use the observables

$$t^l_{23}, \quad t^l_{13}, \quad t^l_{12}, \quad \frac{|m_{\rm sol}|}{|m_{\rm atm}|}, \quad m_{\nu_3}, \tag{9.9}$$

where we relate t^l_{23} to the atmospheric mixing, t^l_{12} to the solar mixing and t^l_{13} to the reactor mixing. In this case we are going to be able to fit just five parameters. For this reason and because the uncertainties in the above observables are significantly bigger than the uncertainties in the quark sector, the fits of the coefficients of the neutrino mass matrix have large errors and they may leave a room for other solutions once the experimental uncertainties improve. Since we only have an upper bound for the reactor angle, t^l_{13}, we fit the solutions in the neighborhood of this upper bound.

9.3 Results of the fits

9.3.1 Fit 1: SU(5) ($u = v = 0$) example satisfying the GST relation

This is a SU(5) type solution, and hence $u = v = 0$, which satisfies the GST relation. The textures are as laid out in eq. (9.2).

Quark sector. We can use the expressions eq. (4.2) and eq. (4.4) adapted to the solution of eq. (6.9) in order to fit the Yukawa coefficients, along with the appropriate phases entering into the expressions of mixings. The expansion parameter ε is determined with the Fayet-Iliopoulos term and the appropriate charges cancelling the anomalies, for this case its value is $\varepsilon = 0.183$. The parameters that we fit are the real parameters

$$a^u_{12}\,,\quad a^u_{23}\,,\quad a^d_{22}\,,\quad a^d_{12}\,,\quad a^d_{13}\,,\quad a^d_{23}\,,\quad a^d_{32}\,,\quad \cos(\Phi_2)\,, \tag{9.10}$$

which enter in the expressions of mixings and masses, eq. (4.2)–eq. (4.5). Note that in these expressions the coefficients a^f_{ij} can be complex but for the fit we choose them real and write down explicitly the phases. We are free to choose the parameters to fit. However we need to check which are the most relevant parameters to test the symmetry. Thus we follow this as a guideline to choose the parameters to fit and leave other parameters fixed. Due to the form of eq. (6.9) the mixing angles in the $(2,3)$ sector of both matrices contribute at the same order in the $V_{\rm CKM}$ matrix mixing, $s^Q_{23} = |a^d_{23} - a^u_{23} e^{i\Phi_{X_{23}}}|\varepsilon^2$, so we have decided to put a phase here. In the s^u_{12} diagonalization angle and the second eigenvalue of Y^u the combination $a^u_{22}e^{i\Phi_3} - {a^u_{23}}^2$ appears, so we have chosen as well to include a phase difference there. The fixed parameters are then

$$a^u_{22}\,,\qquad \Phi_1\,,\qquad \Phi_3\,,\qquad \Phi_{X_{23}}\,, \tag{9.11}$$

where Φ_1 has the form of eq. (4.7) and the phases Φ_3 and $\Phi_{X_{23}}$ can be written as[9]

$$\Phi_3 = \phi^u_{22} - 2\phi^u_{23}\,,\qquad \Phi_{X_{23}} = (\phi^d_{33} - \phi^d_{23}) - (\phi^u_{33} - \phi^u_{23})\,. \tag{9.12}$$

The results of the fit in the quark sector appear in the second column of table 10. Given these results we can think that the structure of Yukawa matrices has the following form

$$Y^u = \begin{bmatrix} * & y_{12}e^{i\Phi_1} & y_{13} \\ y_{12}e^{i\Phi_1} & y_{22}e^{i\Phi_3} & y_{23} \\ y_{13} & y_{23} & 1 \end{bmatrix},\qquad Y^d = \begin{bmatrix} * & y_{12}e^{i\Phi_2} & y_{13}e^{i\Phi_2} \\ \left[y_{21}e^{i\Phi^R_2}\right] & y_{22} & y_{23} \\ * & y_{32} & 1 \end{bmatrix}, \tag{9.13}$$

where y_{ij} denote real elements and we have associated the phases Φ_i to particular elements of the matrices. Note that we need three phases to determine the amount of CP violation experimentally required because in all the fits we found $\Phi_{X_{23}} = 0$. If this phase was not zero then it could have been associated to the Y^d_{23} element. The entries marked with $*$ cannot be determined because they are not restricted by masses and mixings, due to the structure of the Yukawa matrices. The value of $y_{21}e^{i\Phi^R_2}$ is determined indirectly because we need to satisfy the GST relation so $t^R_{12} = t^L_{12}$ for both up and quark sectors.

Lepton sector. We have fixed the coefficients of Y^d in the quark sector and now we can use the results for the charged lepton matrix Y^e. The masses of the charged lepton are

[9] In terms of the β_i phases appearing in the diagonalization matrices, eq. (A.1), we have $\Phi_1 = -\beta^{u\ L}_3$, $\Phi_2 = -\beta^{d\ L}_3$ and $\Phi_{X_{23}} = (\beta^{d\ L}_2 - \beta^{d\ L}_1) - (\beta^{u\ L}_2 - \beta^{u\ L}_1)$.

Quark Fitted Parameters			
GST sol. 2		GST sol. 2, $u,v\neq 0$	GST sol. 3, $u,v\neq 0$
Parameter	BFP Value	BFP Value	BFP Value
a^u_{12}	2.74 ± 0.61	1.04 ± 0.19	2.74 ± 0.71
a^u_{23}	1.68 ± 0.17	1.34 ± 0.13	1.41 ± 0.18
a^d_{22}	1.08 ± 0.18	1.05 ± 0.11	0.70 ± 0.23
a^d_{12}	0.93 ± 0.15	0.55 ± 0.20	0.74 ± 0.13
a^d_{13}	0.29 ± 0.21	0.30 ± 0.14	0.74 ± 0.17
a^d_{23}	0.79 ± 0.10	0.70 ± 0.13	0.66 ± 0.35
a^d_{32}	0.48 ± 0.17	1.28 ± 0.32	1.28 ± 0.58
$\cos(\Phi_2)$	0.454 ± 0.041	0.456 ± 0.041	0.547 ± 0.424
Quark Fixed Parameters			
ε	0.183	0.217	0.154
a^u_{22}	1	1	1.4
$\cos(\Phi_3)$	0.8	0.83	0.8
$\cos(\Phi_{X_{23}})$	1	1	1
Φ_1	$\pi/2$	$\pi/2$	$\pi/2$
χ^2			
χ^2	1.47	2.41	4.32

Table 10: Quark fitted parameters for the examples of section 6). The second column corresponds to the Solution 2 in the SU(5) ($u=v=0$) case, the third column to the Solution 2 in the $u\neq -v\neq 0$ case. The fourth column presents the fit to the Solution 3 in the $u\neq -v\neq 0$ case.

obtained through the SU(5) relations, ensuring the correct value of charged lepton masses, once the masses of the d-quarks are in agreement with experimental information. Thus in this case we perform a fit just for coefficients of the neutrino mass matrix, Y^ν, using the ratio of neutrino mass differences (solar to atmospheric), the mass of the heaviest neutrino and the lepton mixings, which have a contributions from both the charged leptons and the neutrinos. Here the relevant parameter that we need from the quark sector is a^d_{32} because the tangent of the angle diagonalizing Y^e on the left is related to this parameter: $t^e_{23}=a^e_{23}\propto a^d_{32}$. Since this is an $O(1)$ mixing we have to take it into account for the results of the U_{MNS} mixings, thus we have

$$t^l_{23}=\frac{|c^e_{23}s^\nu_{23}e^{-i\phi_{X_{23}}}-s^e_{23}c^\nu_{23}|}{|s^\nu_{23}s^e_{23}+c^\nu_{23}c^e_{23}e^{i\phi_{X_{23}}}|}\,, \tag{9.14}$$

where we use the expression eq. (5.7) to determine s^ν_{23} and c^ν_{23}, and the approximation $t^e_{23}=a^d_{32}$; $\phi_{X_{23}}$ is a phase relating e and ν mixings in the $(2,3)$ sector [38]. We denote the U_{MNS} angles by the superscript l and by e and ν the charged lepton and neutrino mixings respectively. The mixings t^l_{13} and t^ν_{12} are essentially given by the neutrino mixings, eqs. (6.28), so we fit these mixings according to eq. (5.8) and eq. (5.9) respectively. We note from table 11 that in the lepton sector we need two phases, $\phi_{X_{23}}$ and ϕ^ν. The phase $\phi_{X_{23}}$ can be associated to the charged lepton sector and we can put it in the Y^e_{23} entry. The

Neutrino Fitted Parameters						
Parameter	GST sol. 2		GST sol. 2, $u, v \neq 0$		GST sol. 3, $u, v \neq 0$	
	M_P	M_G	M_P	M_G	M_P	M_G
	BFP value	BFP value	BFP value	BFP value	BFP value	BFP value
a^{ν}_{23}	0.75 ± 0.79	0.67 ± 0.61	0.21 ± 0.25	0.85 ± 0.27	0.30 ± 0.18	0.40 ± 0.15
a^{ν}_{13}	1.41 ± 1.32	1.36 ± 1.10	0.97 ± 0.47	1.25 ± 0.63	1.02 ± 0.50	1.45 ± 0.70
a^{ν}_{12}	2.23 ± 0.92	2.10 ± 0.81	1.25 ± 0.29	2.08 ± 0.69	1.35 ± 0.34	1.97 ± 0.45
a^{ν}_{22}	1.84 ± 1.37	1.96 ± 1.92	1.23 ± 1.41	1.98 ± 0.79	1.48 ± 1.31	2.26 ± 1.6
a^{ν}_{32}	1.47 ± 1.93	0.98 ± 0.91	0.65 ± 0.70	1.53 ± 0.75	0.53 ± 0.78	0.56 ± 0.98
Neutrino Fixed Parameters						
ε	0.183		0.217		0.154	
a^{e}_{23}	$a^{d}_{32} = 0.48$		-1.6		1.2	
a^{ν}_{33}	1		1		0.7	1
σ	29/2	21/2	29/2	19/2	55/4	39/4
$c(\phi_{X_{23}})$	0.29		0.29		1	0.5
$c(\phi^{\nu})$	-1		-0.5		0.86	1
(n_2, n_3)	$(1/4, 1/2)$		$(1/8, 1/2)$			
χ^2						
χ^2	0.44	0.12	1.67	0.49	2.16	0.72

Table 11: Neutrino fitted parameters for the examples of Section 6. The second column corresponds to the Solution 2 in the SU(5) ($u = v = 0$) case, the third column to the Solution 2 in the $u+v \neq 0$ case. The fourth column presents the fit to the Solution 3 in the $u + v \neq 0$ case. Here $c(y)$ is the cosine of the respective parameter.

second phase, ϕ^{ν} can be assigned to Y^{ν}_{22}. We fit the mass ratio and the heaviest neutrino state using their expressions appearing in eqs. (6.28). The results for this fit appear in the second column of table 11.

9.3.2 Fit 2 and Fit 3: Extended SU(5) solutions with $u + v \neq 0$ satisfying the GST relation

These are both extended SU(5) solutions, with $u + v \neq 0$, satisfying the GST relation. Fit 2 corresponds to the textures laid out in eq. (9.3), and Fit 3 corresponds to the textures laid out in eq. (9.4).

Quark sector. This section is completely analogous to the previous one, the only difference is in the value of ε. We present here two examples. The first example corresponds to the first solution of eq. (6.9), which we called Solution 2, and corresponds to $\varepsilon = 0.217$ according to the charges of the third row of eq. (6). The second example corresponds to the first solution of eq. (6.11), which has been called Solution 3 and corresponds to $\varepsilon = 0.154$, according to the charges of the fourth row of eq. (6). The fitted and fixed parameters are also those of the previous example, eq. (9.10) and eq. (9.11) respectively. The results for the quark fitting are presented in the third and fourth column of table 10, respectively, so we can compare directly with the previous case.

Lepton sector. This case is different from the section 9.1 because now we do not have the SU(5) relations. Instead the parameter k_e is different from k_d, as explained in section 4, and hence $Y^e \neq (Y^d)^T$. In this case we perform two fits, one for the coefficients of the charged lepton mass matrix, Y^e and another for the coefficients of the neutrino mass matrix, Y^ν.

For the Solution 2, taking into account the value of the charges, the second row of table 6, and that $m = u + v = 1/2$ we have $k_e = -8/3$. We note in this case that since we need $m_b \approx m_\tau$, which are given by

$$\begin{aligned} m_b &= m_t \varepsilon^{|k_d|}\,, \qquad k_d = \frac{l_3 + 3e_3 + u + 4(u+v)}{3} \\ m_\tau &= m_t \varepsilon^{|k_e|}\,, \qquad k_e = l_3 + 3e_3 + u + (u+v)\,, \end{aligned} \tag{9.15}$$

we expect the sum $(u + v)$ to remain small.

Now the coefficients a^e_{23} and a^d_{32} are not related but we can fix a^e_{23} in the neutrino sector such that it is in agreement with the results from neutrino oscillation. We have performed a fit using the experimental information of the parameters of eq. (9.9). Here we have also used the expression eq. (9.14) in order to fit the atmospheric angle, the expressions eq. (5.8) and eq. (5.9) to fit t^l_{13} and t^l_{12} (reactor and solar angle respectively) and the mass ratio and the heaviest neutrino state using their expressions appearing in eqs. (6.28). The results for this fit appear in the third column of table 11.

Once the parameter a^e_{23} has been fixed we fit the parameters of the charged lepton mass matrix, of the form eq. (6.32) and the other parameters as in the first solution of eq. (6.9). In this case the relevant parameters are a^e_{12} and a^e_{22}. However if we just fit the expressions

$$\begin{aligned} \frac{m_e}{m_\mu} &= \frac{|a^e_{12}|^2}{|(a^e_{22} - a^e_{23}a^e_{32})|^2}\varepsilon^{4/3} = {s^e_{12}}^2\,, \\ \frac{m_\mu}{m_\tau} &= (a^e_{22} - a^e_{23}a^e_{32})\varepsilon^2\,, \end{aligned} \tag{9.16}$$

the coefficients a^e_{12} and a^e_{22} are not quite $O(1)$ so we have to make use of a coefficient, c such that $(a^e_{22} - a^e_{23}a^e_{32}) \rightarrow\ (a^e_{22} - a^e_{23}a^e_{32})/c$, e.g. $c = 3$, in order to have acceptable values for charged lepton masses. This fit is presented in the second column of table 12. In this case the extra-coefficient needed for the fit is not really justified in the context of just a single U(1) symmetry.

For the Solution 3, we have $m = 1/2$, $k_e = -13/6$, according to the charges of the third row of table 6. The fit of the coefficients of the neutrino mass matrix are completely analogous for Solution 2 and they appear in the third column of table 11. The relevant parameters for the charged lepton sector are

$$\begin{aligned} \frac{m_e}{m_\mu} &= \frac{|a^e_{12}|^2}{|(a^e_{22} - a^e_{23}a^e_{32})|^2}\varepsilon^{29/6} = {s^e_{12}}^2\,, \\ \frac{m_\mu}{m_\tau} &= (a^e_{22} - a^e_{23}a^e_{32})\varepsilon^{7/4}\,. \end{aligned} \tag{9.17}$$

For this case *there is no need* to invoke another coefficient as for the Solution 2. $O(1)$ coefficients in this case can account for the masses and mixings in the leptonic sector. Once the coefficient a^e_{23} is fitted in the charged lepton sector then we need to use this

Charged lepton Fitted Parameters		
	GST sol. 2, $u, v \neq 0$	GST sol. 3, $u, v \neq 0$
Parameter	BFP Value	BFP Value
a^e_{12}	0.56 ± 0.006	2.88 ± 0.032
a^e_{22}	0.92 ± 0.013	1.87 ± 0.013
Charged lepton Fixed Parameters		
ε	0.217	0.154
a^e_{23}	-1.6	1.2
a^e_{32}	1.8	1.2
χ^2		
χ^2	0.05	1.2×10^{-5}

Table 12: Charged lepton fitted parameters for the examples of section 6 The second column corresponds to the Solution 2 in the $u + v \neq 0$ case. The fourth column presents the fit to the Solution 3 in the $u + v \neq 0$ case.

parameter as a fixed parameter in the fit for the neutrino sector but in this case the fit is not as good as for the previous solution. The results are presented in the third column of table 12.

9.3.3 Fit 4: SU(5) type ($u = v = 0$) solution not satisfying the GST relation

This is a SU(5) type solution, hence $u = v = 0$, which doesn't satisfy the GST relation. The charges are as laid out in eq. (9.5).

Quark sector. Here we also use the expressions eq. (4.2) and eq. (4.4) adapted to the solution eq. (7.3) for $r'_d = l_1 - l_3 = 1$ and check the fit with an exact numerical solution, which agrees with the fit to eq. (4.2) and eq. (4.4) within a 5% error. Since the fit can just fit eight parameters, in this case it is not possible to select out "the best fit", according to the criteria that we have used for the previous fits, so we present the following solution for the coefficients of the up and down Yukawa matrices:

$$a^u = \begin{bmatrix} 0.42 & 0.58e^{-i\pi/2} & 0.51 \\ 0.58e^{-i\pi/2} & 0.9e^{-i\pi} & 0.43e^{-i\pi/2} \\ 0.51 & 0.43e^{-i\pi/2} & 1 \end{bmatrix},$$
$$a^d = \begin{bmatrix} e^{-i0.5} & 0.8 & 0.29e^{i0.48} \\ 1.63e^{-i1.49} & 0.86e^{-i1.2} & 0.55e^{-i0.7} \\ e^{-i0.79} & 0.4e^{-i0.5} & e^{-i3.05} \end{bmatrix}. \qquad (9.18)$$

For this fit we have $\chi^2 = 2.31$.

Lepton sector. In the lepton sector, once we have done the fit to the quark masses, the SU(5) relations produce acceptable values for the charged lepton masses, what we need to care about are the mixings for the neutrino sector. According to the expressions for the mixings in the (1, 2) and (1, 3) neutrino sector, eq. (7.14), now $t^\nu_{13} = a^\nu_{13}\varepsilon/\sqrt{{a^\nu_{33}}^2 + {a^\nu_{23}}^2}$

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and $t^\nu_{12} = a^\nu_{12}\varepsilon^{1/4}/(c^\nu_{23}a^\nu_{22} - s^\nu_{23}a^\nu_{32}))$, for $(n_2, n_3) = (-3/8, 0)$. On the other hand the mixings in the charged lepton sector go as $t^e_{12} = |a^d_{21} + 3a^d_{23}a^d_{31}/a^d_{33}|\varepsilon/3|a^d_{22} + 3a^d_{32}a^d_{23}|$ and $t^e_{13} = a^d_{31}\varepsilon/|a^d_{33} + |a^d_{32}|^2|$, so here these contributions are important to the U_{MNS} s^l_{12} and s^l_{13} mixings, identified respectively to the solar and reactor mixings, for example for s^l_{13} we have

$$\begin{aligned} s^l_{13} = |&c^e_{12}c^e_{13}s^\nu_{13} - c^\nu_{13}(e^{i(\beta^e_1-\beta^\nu_1)}c^\nu_{23}(c^e_{12}c^e_{23}s^e_{13} + e^{i\beta^e_3}s^e_{12}s^e_{23}) - \\ &- e^{i(\beta^e_2-\beta^\nu_2)}s^\nu_{23}(e^{i\beta^e_3}s^e_{12}c^e_{13} - c^e_{12}s^e_{13})s^e_{23})| \,. \end{aligned} \tag{9.19}$$

The mixing s^l_{23} is driven by the neutrino mixing s^ν_{23}

$$s^l_{23}c^l_{13} \approx |e^{i(\beta^e_2-\beta^\nu_2)}s^\nu_{23}c^e_{12}c^e_{23} - e^{i(\beta^e_1-\beta^\nu_1)}s^e_{23}c^\nu_{13}c^\nu_{23}| \,. \tag{9.20}$$

Despite all the contributions to the mixings s^l_{13} and s^l_{12} we can reproduce the observed masses and mixings in the neutrino sector with $O(1)$ coefficients and with out any phase in this sector, we just use the phases of the right handed quark matrix, which are given by

$$\begin{aligned} \beta^e_1 &= \mathrm{ArcTan}\left[\frac{\sin(\phi^d_{33})}{\cos(\phi^d_{33}) + |a^d_{32}|^2}\right] - \phi^d_{31} \\ \beta^e_2 &= (\phi^d_{32} - \phi^d_{33}) + \beta^{dR}_1 \\ \beta^e_3 &= (\phi^d_{22} - \phi^d_{21}) - \beta^{dR}_2 \,, \end{aligned} \tag{9.21}$$

and are specified in eq. (9.18). The results of this fit are given in the second row of in table 13.

9.3.4 Fit 5: SU(5) type ($u = v = 0$) solution not satisfying the GST relation

This is a SU(5) type solution, and hence $u = v = 0$ which doesn't satisfy the GST relation. The textures are as laid out in eq. (9.6).

Quark sector. Here we present the following solution for the case $r_d = l_1 - l_3 = \frac{3}{2}$, in this case the coefficients of the up and down Yukawa matrices:

$$\begin{aligned} a^u &= \begin{bmatrix} 0.5 & 0.6e^{-i\pi/2} & 0.5 \\ 0.6e^{-i\pi/2} & e^{-i\pi} & 0.43e^{-i\pi/2} \\ 0.5 & 0.43e^{-i\pi/2} & 1 \end{bmatrix}, \\ a^d &= \begin{bmatrix} 1 & 0.72 & 0.29e^{i0.49} \\ 1.82e^{-i2.28} & 0.76e^{-i1.12} & 0.55e^{-i0.71} \\ e^{-i1.57} & 0.4e^{-i0.41} & e^{-i2.951} \end{bmatrix}. \end{aligned} \tag{9.22}$$

For this fit we have $\chi^2 = 2.10$.

Lepton sector. The analysis of this fit is completely analogous to the Fit 4, the results of the fitting procedure is presented in the second column of table 13.

Neutrino Fitted Parameters		
Non GST sol. 1		Non GST sol. 2
Parameter	BFP Value	BFP Value
a^{ν}_{23}	1.6 ± 0.8	2 ± 0.9
a^{ν}_{13}	1.4 ± 0.7	0.9 ± 0.3
a^{ν}_{12}	1 ± 0.6	1.6 ± 0.3
a^{ν}_{22}	0.67 ± 0.27	0.5 ± 0.4
Neutrino Fixed Parameters		
ε	0.19	0.185
a^{e}_{23}	$-3a^{d}_{32} = -1.2$	$-3a^{d}_{32} = -1.25$
a^{ν}_{33}	1	1
a^{N}_{22}	2	2
σ	$(4.5, 0.5)$	$(5, 1)$
(n_2, n_3)	$(-3/8, 0)$	$(-5/8, -1/4)$
χ^2		
χ^2	$(5.09, 4.77)$	$(4.78, 3.79)$

Table 13: Neutrino fitted parameters for two of the non GST examples of Section 7 The second and third columns correspond respectively to solution 1 and 2 in the non GST SU(5) ($u = v = 0$) cases, for the first one we have used $r'_d = 1$ and for the second $r'_d = 3/2$. While we have fitted in the first case t^{ν}_{13} to saturate its current upper limit, we have allowed for the second case to be smaller than it. The first entry for σ corresponds to the fit using M_P and the second entry using M_G; analogously for χ^2.

$\tan\beta$, a^u_{33} and a^d_{33}			
Parameter	GST sol. 2, $u = v = 0$	GST sol. 2, $u, v \neq 0$	GST sol. 3, $u, v \neq 0$
a^u_{33}	$(1, 1.34)$	$(1.1.3)$	$(1, 1.3)$
a^d_{33}	$(5.33^{-1.13}_{+2.81}, 2.40^{-0.12}_{+0.13})$	$(3.49^{-0.73}_{+1.84}, 1.62^{-0.09}_{+0.08})$	$(3.23^{-0.68}_{+1.70}, 1.62^{-0.09}_{+0.10})$
$\tan\beta$	$(3.00^{-0.66}_{+4.82}, 1.00^{-0.06}_{+0.06})$	$(3.00^{-0.66}_{+1.61}, 1.07^{-0.07}_{+0.07})$	$(3.00^{-0.45}_{+1.61}, 1.07^{-0.07}_{+0.07})$
$(\epsilon,\ \lvert k_d\rvert)$	(0.183, 5/2)	(0.217, 5/2)	(0.154, 2)
Parameter	Non GST sol. 1, $u = v = 0$	Non GST sol. 2, $u = v = 0$	
a^u_{33}	$(1, 1.2)$	$(1, 1.2)$	
a^d_{33}	$(2.12^{-0.44}_{+0.98}, 1.1^{-0.03}_{+0.08})$	$(2.11^{-0.45}_{+0.87}, 1.3^{-0.06}_{+0.09})$	
$\tan\beta$	$(3^{-0.66}_{+1.61}, 1.3^{-0.1}_{+0.1})$	$(3^{-0.78}_{+1.32}, 1.2^{-0.2}_{+0.2})$	
$(\epsilon,\ \lvert k_d\rvert)$	(0.19, 2)	(0.185, 2)	

Table 14: Value of a^d_{33} and $\tan\beta$ for the different models presented, once a^u_{33} is fixed using m_t.

9.3.5 Top and bottom masses and $\tan\beta$

For these cases $\tan\beta$ and a^d_{33} are a prediction, once the coefficient a^u_{33} is fixed through the value of m_t, $m_t = Y^u_{33} v/\sqrt{2}$. The values of a^u_{33}, a^d_{33} and $\tan\beta$ for the cases presented in this section are given in table 14. We can see that for a natural value of $a^u_{33} = 1$ we have acceptable values for $\tan\beta$ (which should be > 2) and a^d_{33} in any of the cases presented.

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Quark fitted Parameters, SU(3)-like case		
Parameter	BFP Value $\pm\sigma$	BFP Value $\pm\sigma$
$a'u_{22}$	1.11 ± 0.55	1.11 ± 0.07
a^d_{12}	0.66 ± 0.32	2.45 ± 0.20
a^d_{13}	0.10 ± 0.12	0.91 ± 0.15
a^d_{22}	0.74 ± 0.10	1.77 ± 0.09
a^d_{23}	0.45 ± 0.29	1.18 ± 0.12
ϵ^u	0.05 ± 0.007	0.05 ± 0.007
ϵ^d	0.25 ± 0.03	0.16 ± 0.02
$\cos(\Phi_2)$	0.516 ± 0.1	0.450 ± 0.045
Quark Fixed Parameters, SU(3)-like case		
Φ_1^*	$-1.25 \approx= -0.8\pi/2$	$1.120 \approx 0.7\pi/2$
χ^2		
χ^2	0.972	0.974

Table 15: Fitted and fixed parameters for the SU(3)-like case.

9.4 Comparison to the SU(3) case

In this section we present the comparison to a generic SU(3) case [6]. What we fit are the $O(1)$ coefficients of a Yukawa matrices of the form

$$Y^f = \begin{bmatrix} \varepsilon_f^8 & \varepsilon_f^3 & \varepsilon_f^3 \\ \varepsilon_f^3 & \varepsilon_f^2 & \varepsilon_f^2 \\ \varepsilon_f^3 & \varepsilon_f^2 & 1 \end{bmatrix}, \tag{9.23}$$

where we allow two different expansion paramaters ε_u and ε_d and complex phases to reproduce the CP violation phase. It is enough to consider one different phase in each of the Y^u and Y^d matrices. Here we put the phases on Y^d_{13} and Y^u_{12} [20], but we have the freedom to use other choices. We have used here as well the method of minimization that we have used for the U(1) cases. The results of these fits are consistent with previous determination of these parameters, [19, 20], taking into account the change induced by the change of the value used here for the parameter $m_c/m_s = 15.5 \pm 3.7$ and the different methods used for the determination of coefficients.[10] The fits presented here are the fits with the lowest possible χ^2 because of the minimization procedure. We have not included here for the SU(3) case a fit in the neutrino sector because in the SU(3)-like cases the neutrino sector requires more assumptions than in the analogous U(1) cases.

Another important difference between the SU(3) and the U(1) cases presented here is that in the first one there are two parameter expansions ε^u and ε^d which have been fitted while in the U(1) cases there is only one expansion parameter which can be fixed by relating the U(1) symmetry to the cancellation of anomalies and the Fayet-Iliopoulos term. This has allowed that more $O(1)$ coefficients have been able to be fitted.

[10] In [19, 20] $m_c/m_s = 9.5 \pm 1.7$.

Comparison			
	U(1) (GST)	U(1) (Non-GST)	SU(3)-like
# of expansion pars.	1	1	2
# of free pars.(quark sector)	12	>18	10
GST relation	yes	no	yes
prediction for $\tan\beta$	small	small	no
lepton sector	o.k.	o.k	o.k
simple flavour charges	no	yes	yes

Table 16: Some criteria of comparison. Here the number of free parameters corresponds to the number of coefficients, phases and parameter expansions that need to be adjusted or determined in the fits.

By comparing tables 10 and 15 we can see that according to the minimization procedure and the criteria of $O(1)$ coefficients, the second case of the SU(3) solution fits better the data. However the U(1) solutions also have a good fit and taking into account the fact that for the neutrino sector we just have added the SRHND conditions, the fits in both of the U(1) cases presented are good. We can therefore consider that U(1) symmetries are still an appealing description of the fermion masses and mixings observed. Note that although the Solution 3 in the $u \neq v \neq 0$ does not fit the data as well as the Solution 2 (in either case, $u = v = 0$ or not) in the quark sector, it does reproduce masses and mixings in the charged lepton sector. We have for this case $Y^e \neq (Y^d)^T$ but we have $m_b \approx m_\tau$ without introducing ad-hoc $O(1)$ coefficients in order to reproduce the appropriate mixings.

Given the results of these fits we need further criteria in order to compare models based in anomalous U(1) models and non-abelian models, such as SU(3). These other criteria may be found in the predictions that the models presented here can give in the supersymmetric sector.

10. Flavour issues in SUSY flavour symmetry models

Since the flavour symmetry is expected to be broken at a high energy scale, non supersymmetric models will have a hierarchy problem, since the cutoff of the theory must at least be of the order of the flavour symmetry breaking scale. Supersymmetric models with soft breaking parameters around the TeV scale do not have this problem. For this reason flavour symmetries are almost exclusively considered in the context of one of the minimal supersymmetric models, or one of the popular SUSY unified theories. The soft lagrangian parameters are strongly constrained by the supersymmetric flavour problem and the supersymmetric CP problem.

The supersymmetric flavour problem needs the soft scalar mass squared matrices to be diagonal to good approximation at high energy scales, since the off-diagonal elements contribute to one-loop flavour violating decays such as the highly constrained $\mu \to e\gamma$ in the lepton sector and $b \to s\gamma$ in the quark sector. It also requires that the trilinear couplings are aligned well to the corresponding Yukawa matrix, since off diagonal elements in the trilinears in the mass eigenstate basis also contribute to highly constrained decays.

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The supersymmetric CP problem is related to the phases of the parameters in the soft lagrangian. The general requirement is that these phases need to be small for the majority of soft breaking parameters.

The reason that these problems are relevant in the context of family symmetries is that in general, the existence of the family symmetry and the fields that break it can give dangerous contributions to the soft lagrangian parameters. It would be remiss to look at these models but not check whether CP violation or flavour violation is likely to rule them out. The starting point for investigating these problems is to consider the hidden sector part of the theory, which leads to the size and phases of the vevs of the fields which break the U(1) symmetry, θ and $\overline{\theta}$.

10.1 The flavon sector

We start by considering the values of the expansion parameters ϵ and $\overline{\epsilon}$. They are defined by:

$$\epsilon \equiv \frac{\langle\theta\rangle}{M} \qquad \overline{\epsilon} = \frac{\langle\overline{\theta}\rangle}{M}\,, \tag{10.1}$$

where θ and $\overline{\theta}$ are scalars which break the $U(1)_F$ symmetry, and have charges of $1, -1$ respectively under the symmetry. The solutions that we have found in the previous sections need to have the charges of θ different from 1 and -1 but the present discussion is in order to explain how the value of ϵ can be fixed. We wish to arrange that $\epsilon = \overline{\epsilon}$, which entails arranging that the potential is minimized by $\langle\theta\rangle = \langle\overline{\theta}\rangle$. This would be simple if the U(1) were non-anomalous, and thus missing a Fayet-Iliopoulis term. If we set the θ sector of the superpotential to be:

$$W_\theta = S(\theta\overline{\theta} - M_\theta^2)\,. \tag{10.2}$$

We introduce a new field, X, which has charge q_X under U(1). q_X will be unspecified, but some number such that when $\langle X\rangle \neq 0$, it doesn't contribute to the fermion mass operators (or, at the very least, it doesn't contribute at leading order). Then, if we give θ and $\overline{\theta}$ the same soft mass,[11] and require that X doesn't get a soft mass, we end up with a hidden sector potential:

$$V = |\theta\overline{\theta} - M_\theta^2|^2 + \frac{g^2}{2}\left(|\theta|^2 + |\overline{\theta}|^2 - q_X|X|^2 + \xi^2\right)^2 + m^2(\theta^2 + \overline{\theta}^2)\,. \tag{10.3}$$

If we minimize this potential with respect to $\theta, \overline{\theta}$ and X, we end up with the following constraints:

$$\frac{\partial V}{\partial\theta} = 0 = 2\overline{\theta}(\theta\overline{\theta} - M_\theta^2) + g^2\theta(|\theta|^2 - |\overline{\theta}|^2 + q_X|X|^2 + \xi^2) + 2m^2\theta \tag{10.4}$$

$$\frac{\partial V}{\partial\overline{\theta}} = 0 = 2\theta(\theta\overline{\theta} - M_\theta^2) + g^2\overline{\theta}(|\theta|^2 - |\overline{\theta}|^2 + q_X|X|^2 + \xi^2) + 2m^2\overline{\theta} \tag{10.5}$$

$$\frac{\partial V}{\partial X} = 0 = \frac{g^2}{2}2X(|\theta|^2 - |\overline{\theta}|^2 + q_X|X|^2 + \xi^2)\,. \tag{10.6}$$

[11]This requirement may seem somewhat strong, but we also wish to minimize flavour violation coming from the D-term associated with U(1), which is proportional to $m_\theta^2 - m_{\overline{\theta}}^2$, and will provide a non-universal contribution to the scalar masses. This contribution will lead to off diagonal elements in the SCKM basis which can easily be dangerously large with regard to flavour violation.

Since X doesn't have a mass term, it would be massless unless $\langle X \rangle \neq 0$. Therefore, of the two solutions of eq. (10.6), we have to take $X \neq 0$. From this, we see that:

$$|\theta|^2 - |\overline{\theta}|^2 + q_X |X|^2 + \xi^2 = 0 \,. \tag{10.7}$$

Substituting eq. (10.7) into eq. (10.4) and eq. (10.5), and multiplying by $\overline{\theta}$ and θ respectively, we find:

$$0 = \theta\overline{\theta}(\theta\overline{\theta} - M_\theta^2) + m^2\theta^2 \tag{10.8}$$

$$0 = \theta\overline{\theta}(\theta\overline{\theta} - M_\theta^2) + m^2\overline{\theta}^2 \,. \tag{10.9}$$

From this, we can deduce that either $\theta = \overline{\theta} = 0$ or $|\theta| = |\overline{\theta}| = M_\theta$. The potential is minimized by the second solution if $m^2 < 2M_\theta^2$. As we expect M_θ to be a GUT scale mass, and m to be a TeV scale soft mass term, we find, that as desired, that we will have:

$$\langle \theta \rangle = \langle \overline{\theta} \rangle \Rightarrow \epsilon = \overline{\epsilon} \,. \tag{10.10}$$

This allows us to consider Yukawa textures without having to keep track of whether the overall charge for each term is positive or negative.

10.1.1 Getting ϵ from the Fayet-Iliopoulos term

The GST requirement leads to needing flavon fields with opposite charges. under $\mathrm{U}(1)_F$. Were this not the case, we would have an elegant way of generating $\langle \theta \rangle$. Consider a simple case where θ doesn't have a superpotential mass term, but does have a soft mass:

$$V = \frac{g^2}{2}(-|\theta|^2 + \xi^2)^2 + m_\theta^2\theta^2 \,. \tag{10.11}$$

Then, without the need for an explicit mass term in the superpotential, we would find that minimizing the potential with respect to θ would lead to:

$$\langle \theta \rangle = \xi\sqrt{1 + \frac{m_\theta^2}{\xi^2}} \approx \xi \,. \tag{10.12}$$

Where the final approximation is due to the fact that we expect ξ^2 to be much larger than m_θ^2. So we have managed to set $\langle \theta \rangle$ from ξ, which can be predicted from string theory. So this allows one to predict the flavon vev, rather than having to put it in by hand.

This provides a motivation for trying to set up the case where $\langle \theta \rangle$ and $\langle \overline{\theta} \rangle$ could both be set by the FI term. However, it doesn't seem possible to make this work without adding in either an extra symmetry, or extra matter. Even then, trying to arrange things so that $\langle \theta \rangle = \langle \overline{\theta} \rangle = z\xi$, with z some real number is difficult.

10.2 Yukawa operators

Since the net U(1) charge can be either positive or negative and we have $\epsilon = \overline{\epsilon}$, an effective potential has the following form:

$$W = \sum_{f=u,d;\, ij} Q^i f^{c\,j} H_f a_{ij}^f \epsilon^{|q_i + f_j + h_f|} + \sum_{f=e,n;\, ij} L^i f^{c\,j} H_f a_{ij}^f \epsilon^{|l_i + f_j + h_f|} \,. \tag{10.13}$$

We cannot say anything in particular about the Kähler potential. We can assume that the phases responsible for CP violation only appear in the flavour sector. Then observable CP violating phases will be put into the Yukawa couplings indirectly from the effective superpotential of eq. (10.13). In general we can consider an effective Kähler potential of the form:

$$K = K_o(t_\alpha) - \ln(S + \bar{S} + \delta_{GS}) + \sum_i f_i(t_\alpha)\theta_i\bar{\theta}_i + \cdots + \sum_{ij} K^{\Phi}_{ij}\Phi^i\bar{\Phi}^j \,, \tag{10.14}$$

where K_o is the Kähler potential of the moduli fields, $t_\alpha = T_\alpha + \bar{T}_\alpha$, S is the dilaton, $f_i(t_\alpha)$ are possible functions of these moduli fields e.g. $f(t) = \Pi^p_{\alpha=1} t_\alpha^{n(\alpha)_{ij^*}}$. But we cannot specify the form of the Kähler metric. It may be that the Kähler metric is canonical, in which case $K^{\Phi}_{ij^*} = \delta_{ij^*}$. Such a form has a good change of leading to acceptable phenomenology, since the scalar mass matrices will be proportional to the identity at the appropriate high energy scale. When rotating the scalar mass matrices to the super-CKM (SCKM) basis at the high energy scale, the transformation will leave the mass matrices invariant. Flavour violation tends to be proportional to off-diagonal elements in the scalar mass matrices in the SCKM basis, so any flavour violation will be due to RG effects, and will therefore be suppressed. On the other hand, the Kähler metric could have off-diagonal structure, in which case the risk of flavour violating effects would be high, and the case where the Kähler metric is diagonal but non-universal is potentially very interesting since flavour changing effects are induced in general by the SCKM rotation.

10.3 The SUSY CP problem

10.3.1 The μ problem

In order to avoid the μ problem, a symmetry or other mechanism to protect μ from unwanted contributions needs to be introduced. The μ parameter can have contributions from the superpotential, (expected to be at the Planck scale) and from the Kähler potential, via the Giudice-Masiero mechanism [11] or other mechanisms [21, 22], $\mu = \mu_W + \mu_K$. The charges of the fields H_u and H_d under the flavour symmetry can be chosen in such a way that $\mu_W(M_P)$ is forbidden in the superpotential. Then another field, S can be introduced, so that the term $\lambda S H_u H_d$ is allowed in the Kähler potential, which generates an effective $\mu = O(m_{3/2})$. Note that in the cases that we have found for $u + v \neq 0$ there is no μ_W at M_P. In general for a theory containing two flavon fields with opposite charges, once the flavour symmetry is broken below the Planck scale, the contributions to the μ term are:

$$\epsilon^{|u+v|} H_u H_d \mu_W + \epsilon^{|u+v|} H_u H_d \mu_K \,. \tag{10.15}$$

Thus, even if the μ term is missing from the superpotential at renormalizable level, it will be generated by non-renormalizable operators once the family symmetry is broken. However, it will appear suppressed by a factor of $\epsilon^{|u+v|}$. To get an sufficient suppression, either $|u + v|$ must be large or ϵ must be small. Obviously, since the same factor $\epsilon^{|u+v|}$ appears suppressing both superpotential and Kähler potential μ contributions, there is no extra constraint from considering the second term in eq. (10.15).

However, $|u+v|$ is related to the anomaly cancellation conditions considered in section 2. There are two possibilities for having small $|u+v|$. The first is to have small expansion parameters, ϵ; however if ϵ becomes too small, it makes predicting the fermion mass hierarchy very difficult. The second is to accept a contribution to μ that is larger than order $\mathrm{O}(m_{3/2})$; however phenomenologically, the total μ should not be much bigger than the $O(m_{3/2})$. It is, however, possible to apply a new discrete symmetry to disallow the superpotential μ term, which never allows any flavon corrections to generate it.

10.3.2 Electric dipole moment constraints

The electric dipole moments (EDMs) constrain the form of the trilinear couplings, $(Y_f^A)_{ij}$. The trilinear couplings are defined through $(Y_f^A)_{ij} H_f Q_i f_j^c$. Here we need to ensure that there is not a large contribution from the phases found in the trilinear terms to the CP violating phases. In the context of flavour symmetries it is usually postulated that the only phases appearing in the theory are in the Yukawa couplings and any other phase will enter as a consequence of a dependence in the Yukawa couplings. Then to check if the model gives contribution below the bounds one needs to compare the diagonal elements of the Yukawa couplings with the diagonal elements of the trilinear couplings, in the SCKM basis. The trilinear terms in general can be written as:

$$(\mathbf{Y_f^A})_{ij} = Y_{ij}^f F^a \partial_a \left(\tilde{K} + \ln(K_f^f K_i^i K_j^j) \right) + F^a \partial_a Y_{ij}^f \,. \tag{10.16}$$

We can always write the first term in a "factorisable" form [36], such that if the Yukawa couplings, eq. (10.13), are the only source of CP violation then the first term does not give any contribution at the leading order. For the second term, which involves the derivative in terms of the flavon fields, if the flavon field is the only field with $F^\theta \neq 0$ then the diagonal trilinear couplings in the SCKM basis are real at leading order in the flavon fields [39]. Thus there is not an $O(1)$ contribution to the CP phases from this sector.

One can check this simply by writing the last term of eq. (10.16) in the SCKM basis:

$$(F^a \partial_a (\hat{Y}^f))_{ij}^{\mathrm{SCKM}} = F^a (V_L^\dagger)_{ik} (\partial_a V_L)_{kj} (Y_{\mathrm{Diag}})_{jj} + F^a (\partial_a Y_{\mathrm{Diag}})_{ij} + F^a (Y_{\mathrm{Diag}})_{ii} (\partial_a V_R)_{ir} (V_R^\dagger)_{rj} \,. \tag{10.17}$$

Where $V_L^\dagger$ and $V_R^\dagger$ diagonalize the Yukawa matrix: $Y_{\mathrm{Diag}} = V_L^\dagger Y V_R^\dagger$. The leading term of the eq. (10.17) is the second term and it is at most of order θ. If another field has non-zero F-term, $F^X \neq 0$ then all the quantities appearing in eq. (10.16) can be written as a expansion in X and $\theta/M = \varepsilon$:

$$(Y_{\mathrm{Diag}})_{ii} = (a_{ii} + b_{ii} X) \varepsilon^{p_{ii}} \,. \tag{10.18}$$

We are assuming that only the matter sector in eq. (10.13) has phases leading to CP violation, so the term $b_{ii} X \varepsilon^{p_{ii}}$ is real and hence so is:

$$F^X (\partial_a Y_{\mathrm{Diag}})_{ii} = F^X b_{ii} \theta^{p_{ii}} \,. \tag{10.19}$$

10.4 SUSY flavour problem

In addition to the F term contribution to the soft masses we have to add the D term contributions

$$(M^2)_{ij} = (M^2)_{F\ ij} + (M^2)_{D\ ij}\,. \tag{10.20}$$

If the Kähler metric is diagonal in the basis where the symmetry is broken both contributions are diagonal and proportional to the Kähler metric. For example, consider universal SUGRA: $(M^2)_{F\ ij} = K_{ij} m_o^2$. However, even if we assume that the first term is indeed proportional to the Kähler metric, the D-term will not in general be proportional to the Kähler metric:

$$(M^2)_{D\ ij} = \sum_N g_N X_{N\ \theta_a} K_{ij^*}(\theta_a) m_D^2\,, \qquad m_D^2 = O(m_{3/2}^2) \tag{10.21}$$

The main problems for FC processes for these kind of theories are the contributions to the trilinear couplings from the anomalous D-term contribution to the soft masses [26]. For the last issue there is no real solution so far but one can ameliorate the problem by making all the scalars heavier, which is a simply mass suppression.

In order to study all the possible consequences of models with the superpotential structure of eq. (10.13), we can parameterize the Kähler metric according to the different contributions it may have, assuming a broken underlying symmetry with at least two flavon fields with opposite charges. Once this is done we can then study their consequences. As mentioned earlier, this analysis is beyond the scope of this paper, so we just mention how extreme and dangerous situations may arise and we leave the analysis for a future reference [23]. Some authors have studied possible consequences of flavour models for FC effects but very specific assumptions need to be assumed due to the many unknown supersymmetric parameters [5, 24, 25].

The most strict bound for flavour changing processes is coming from the decay $\mu \to e\,\gamma$ [28]–[29] and given the fact that we have a large mixing angle in the left handed sector of the charged lepton matrices it is crucial to determine under which conditions we can produce a suppressed effect. Also the constraints given by the process $B \to \ \Phi\ K_S$ may select out some of the possibilities presented.

10.4.1 Non minimal sugra and diagonal Kähler metric

Consider, for example, the case for which at the scale at which the flavour symmetry is broken, the Kähler metric is diagonal. For this case, we also want the soft scalar mass matrices diagonal but not proportional to the unit matrix, due to possible different D term contributions. Since the general case it is difficult to handle we consider the case where $M^2_{\tilde{f}\ 1} - M^2_{\tilde{f}\ 2}$ is small and $M^2_{\tilde{f}\ 1} - M^2_{\tilde{f}\ 3} > 0$. In order to estimate the flavour changing processes we need to take into account the effects from renormalization group equations (RGE's) and then at the electroweak scale make the transformation to the basis where the fermions are diagonal. Here we consider the case of leptons, since we are interested in determining δ^l_{ij} and in particular δ^l_{12} which is the most constrained parameter due to $B(\mu \to e\,\gamma)$.

We make an estimation of the contributions from the renormalization β functions in this case, such that at the scale where the dominant right handed neutrino it is decoupled we can write the soft masses as

$$M^2_{\tilde{L}\ ij}(M_Y) \approx M^2_{\tilde{L}\ ij}(M_X) - \frac{1}{16\pi^2}\ln\left(\frac{M_X}{M_Y}\right)\left(\beta^{(1)}_{M^2_{\tilde{L}\ ij}}\right) \tag{10.22}$$

for $M_X = M_{\rm G}$ or $M_{\rm P}$, GUT or Planck scales respectively, and for $M_Y = M_{RR\ 3}$ in this case and considering just one loop corrections. The β functions of $M^2_{\tilde{L}\ ij}$, from M_X to $M_{RR\ 3}$ receive the contributions from the MSSM particles plus the contribution from right-handed neutrinos. At M_3 we then run from that scale to the electroweak symmetry breaking scale with the appropriate β function and matter content. In the case of SNRHD scenario and the form of the Yukawa matrices that we have considered in section 9 we can make the following approximations for the β functions':[12]

$$\begin{aligned}
\left(\beta^{(1)}_{M^2_{\tilde{L}\ ii}}\right)^{MSSM} &\approx 2\left[(m^2_{M^2_{\tilde{L}\ ij}} + m^2_{\tilde{H}_d})\left(|Y_{2i}|^2 + |Y_{3i}|^2\right) + m^2_{\tilde{e}_2}(1+a^2)(|Y_{2i}|^2 + r^2_{\tilde{e}_{23}}|Y_{3i}|^2)\right] - \\
&\quad - 6g_2^2|m_2|^2 - \frac{6}{5}g_1^2|m_1|^2 - \frac{3}{5}g_1^2 S \\
\left(\beta^{(1)}_{M^2_{\tilde{L}\ ij}}\right)^{MSSM} &\approx (2m^2_{\tilde{H}_d} + m^2_{\tilde{L}\ i} + m^2_{\tilde{L}\ j})\left(Y^{e*}_{2i}Y^{e*}_{2j} + Y^{e*}_{3i}Y^{e*}_{3j}\right) + \\
&\quad + 2m^2_{\tilde{e}_2}(1+a^2)\left(Y^{e*}_{2i}Y^{e*}_{2j} + r^2_{\tilde{e}_{23}}Y^{e*}_{3i}Y^{e*}_{3j}\right),
\end{aligned} \tag{10.23}$$

where we have assumed that the trilinear terms can be written as $A^f_{ij} = aY^f_{ij}M^2_{\tilde{e}}$, and $M^2_{\tilde{e}}$ is not necessarily diagonal. The parameter S, defined as $S = m^2_{\tilde{H}_u} - m^2_{\tilde{H}_d} + {\rm Tr}[{\rm M}^2_{\tilde{\rm Q}} - {\rm M}^2_{\tilde{\rm L}} - 2{\rm M}^2_{\tilde{\rm u}} + {\rm M}^2_{\tilde{\rm d}} + {\rm M}^2_{\tilde{\rm e}}]$, does not generate big contributions as long the masses involved remain somewhat degenerate. The β functions generated by the dominant right-handed neutrino can be approximated by

$$\left(\beta^{(1)}_{M^2_{\tilde{L}\ ij}}\right)^{\nu_{M_3}} \approx 2Y^{\nu*}_{3i}Y^{\nu}_{3j}\left[m^2_{\tilde{L}3} + m^2_{\tilde{\nu}3}(1+b^2) + m^2_{\tilde{H}_u}\right]. \tag{10.24}$$

From $M_X = M_3$ to $M_Y = M_{\rm S}$ — the supersymmetry breaking scale —, we consider $(\beta^{(1)}_{M^2_{\tilde{L}\ ij}})^{MSSM}$. For this estimation we ignore the effect from $M_{\rm S}$ down to the electroweak scale. At this scale we then transform the renormalized $M^2_{\tilde{L}}$ in the basis where the charged leptons are diagonal. Since there is a large mixing angle ($s^{e_L}_{23}$) in the left sector of Y^e we are interested here only in estimating $(M^2_{\tilde{L}})_{LL}$. We can use the parameterization of appendix A in order to make this transformation, i.e.

$$Y^f_{\rm diag} = V^{f\dagger}_L Y^f V^f_R\,, \qquad (M^2_{\tilde{L}})'_{LL} = V^{f\dagger}_L M^2_{\tilde{L}} V^f_L\,, \tag{10.25}$$

for $V^f_{L,R}$ as parameterized in eq. (A.1), with the β phases as follow

$$\{\beta^{e_L}_1, \beta^{e_L}_2, \beta^{e_L}_3\} = \{\phi^e_{X_{23}}, 0, 0\}\,, \qquad \phi^e_{X_{23}} = \beta^{e_L}_1 - \beta^{e_L}_2\,. \tag{10.26}$$

[12]For the MSSM see for example [27], when including right handed neutrinos, see for example [28].

Using these approximations, we obtain the following results

$$
\begin{aligned}
(M^2_{\tilde L})'_{12} &= s^{e_L}_{12}(c^{e_L}_{23} m^2_{\tilde L\ 22} - m^2_{\tilde L\ 11}) + (c^{e_L}_{12})^2 e^{-i\beta_{3L}} \times \\
&\quad \times \left(c^{e_L}_{23} e^{-i\beta_{2L}} m^2_{\tilde L\ 12} - 2t_{12} c^{e_L}_{23} s^{e_L}_{23} e^{i\beta_{3L}} \mathrm{Re}\{\mathrm{m}^2_{\tilde L\ 23} \mathrm{e}^{-\mathrm{i}\chi}\} - \mathrm{s}^{\mathrm{e_L}}_{23} \mathrm{e}^{-\mathrm{i}\beta_{1L}} \mathrm{m}^2_{\tilde L\ 13} \right), \\
(m^2_{\tilde L})'_{13} &= c^{e_L}_{23} s^{e_L}_{23} s^{e_L}_{12} e^{i\beta_3 L}(m^2_{\tilde L\ 22} - m^2_{\tilde L\ 33}) + c^{e_L}_{12} c^{e_L}_{23} \times \\
&\quad \times \left(\left(e^{-i\chi} c^{e_L}_{23} t_{12} m^2_{\tilde L\ 23} - e^{i\chi} t_{12} t_{23} s^{e_L}_{23} \beta^*_{m^2_{\tilde L\ 23}} \right) + t_{23} e^{i\chi} m^2_{\tilde L\ 12} + m^2_{\tilde L\ 13} \right), \\
(m^2_{\tilde L})'_{23} &= c^{e_L}_{23} s^{e_L}_{23} e^{i\beta_{3L}} \left(m^2_{\tilde L\ 22} - m^2_{\tilde L\ 33} \right) + \\
&\quad + e^{i\beta_{3L}} c^{e_L}_{12} \left((c^{e_L}_{23})^2 e^{-i\chi} m^2_{\tilde L\ 23} - (s^{e_L}_{23})^2 e^{i\chi} \beta^*_{m^2_{\tilde L\ 23}} \right), \\
(m^2_{\tilde L})'_{11} &= (s^{e_L}_{12})^2 \left((c^{e_L}_{23})^2 m^2_{\tilde L\ 22} + (s^{e_L}_{23})^2 m^2_{\tilde L\ 33} \right) \\
(m^2_{\tilde L})'_{22} &= (c^{e_L}_{12})^2 \left((c^{e_L}_{23})^2 m^2_{\tilde L\ 22} + (s^{e_L}_{23})^2 m^2_{\tilde L\ 33} \right) - (c^{e_L}_{12})^2 c^{e_L}_{23} s^{e_L}_{23} 2\mathrm{Re}\{\mathrm{m}^2_{\tilde L\ 23} \mathrm{e}^{-\mathrm{i}\chi}\} \\
(m^2_{\tilde L})'_{33} &= (c^{e_L}_{13})^2 \left((s^{e_L}_{23})^2 m^2_{\tilde L\ 22} + (c^{e_L}_{23})^2 m^2_{\tilde L\ 33} \right) + c^{e_L}_{23} s^{e_L}_{23} \left(2\mathrm{Re}\{\mathrm{m}^2_{\tilde L\ 23} \mathrm{e}^{-\mathrm{i}\chi}\} \right)
\end{aligned}
\tag{10.27}
$$

here the soft masses $m^2_{\tilde L ij}$ are the soft masses at M_{S}, renormalized from $M_X = M_{\mathrm{G}}, M_{\mathrm{P}}$ down to M_3 with the appropriate contributions from the dominant right handed neutrino, eq. (10.22), and eq. (10.23)–eq. (10.24) and then from M_3 to M_S with the appropriate $\beta^(MSSM)$ functions. Thus we began with a diagonal matrix $M^2_{\tilde L}$ at M_X, then the RGE effects up to the scale where M_3 is decoupled generate a non diagonal matrix which receives more RGE contributions from M_3 to M_S. At electroweak scale we transformed to the basis where charged leptons are diagonal. The mixing angles in this sector can be approximated as

$$
s^{e_L}_{12} = \frac{|(a^e_{12} - t_{32} a^e_{13})|}{|(a^e_{22} - a^e_{32} a^e_{23})| \epsilon^{p^e_{12}}}, \qquad s^{e_L}_{13} = \frac{a^e_{13}}{a^e_{33} \epsilon^{p^e_{13}}}, \qquad s^{e_L}_{23} = \frac{a^e_{23}}{a^e_{33}}. \tag{10.28}
$$

The powers p^e_{ij} for the different solutions presented now correspond to $p^e_{12} = 2/3, 14/3$, $p^e_{13} = 29/12, 71/12$ for Fits 2 and 3 respectively. So in this case we see that we need a big suppression of the element $(m^2_{\tilde l\ L})'_{12}$ in order to be in agreement with the observed bound on $\mu \to\ e\gamma$. In the present example the suppression it is related to a bound on $(m^2_{\tilde L 1} - m^2_{\tilde L 2})$ and a relative big set of soft masses. The results of these estimations are presented in table 17.

As we can see from the results of table 17 the estimation of $|(\delta^l_{LL})^E_{ij}|$ is less dependent on the relation among the original soft mass terms $m^2_{\tilde L i}$ than on the value taken for the average s-lepton mass, which indeed needs to be large. Here we note that this is just an estimation on the conditions that $B(\mu \to e\ \gamma)$ imposes on the soft masses, but with out fully checking whether or not appropriate masses for all the MSSM parameters can be obtained. In the following we consider a numerical investigation in the minimal sugra case.

10.4.2 Numerical Investigation of $B(\mu \to e\ \gamma)$ in minimal sugra

The presence of a right-handed neutrino fields leads to RG lepton flavour violation. Since the masses of the right handed neutrinos are so light for the GST solutions, fits 1-3, we attempted a numerical analysis for all of the fits of section 9 using the same modified version of SOFTSUSY [41] as used in [37].

Estimation of δ_{ij} for the Fit.3 of Section 9.			
Paramter	Ex. I	Ex. II	Ex. III
$m_{\tilde{L}1}$[GeV]	520	520	520
$m_{\tilde{L}2}$[GeV]	530	530	570
$m_{\tilde{L}3}$[GeV]	500	500	230
$m_{\tilde{e}1}$[GeV]	520	520	520
$m_{\tilde{e}2}$[GeV]	530	530	550
$m_{\tilde{e}3}$[GeV]	500	300	300
M_1[GeV]	500	500	500
M_2[GeV]	$2M_1$	700	700
M_{H_d}[GeV]	510	510	510
M_{H_u}[GeV]	510	510	510
M_{S}	1000	1000	1000
$\overline{m}_{\tilde{l}}$	514	486	456
$x = m_{\tilde{\gamma}}^2/m_{\tilde{l}}^2$	0.3		
$\lvert(\delta^l_{LL})^E_{12}\rvert$	4.3×10^{-3}	5.6×10^{-3}	1.4×10^{-3}
$\lvert(\delta^l_{LL})^B_{12}\rvert$	$O(10^{-1})$		$O(10^{-2})$
$\lvert(\delta^l_{LL})^E_{13}\rvert$	1.7×10^{-3}	1.8×10^{-3}	1.9×10^{-2}
$\lvert(\delta^l_{LL})^E_{23}\rvert$	5.7×10^{-2}	6.4×10^{-2}	6.3×10^{-1}
$\lvert(\delta^l_{LL})^B_{23}\rvert$	$O(10^{-1})$		$O(10^{-1})$

Table 17: Estimation of $|\delta^l_{ij}|^E$ in the fit 3 presented for the non minimal sugra example and its comparison to the observed bounds $|\delta^l_{ij}|^E$ [28]–[29].

In order to get a good handle, we have embedded the flavour model fits into a string-inspired mSUGRA type scenario, with no D-term contribution to the scalar masses. This scenario was chosen because it is expected to be the embedding with the lowest flavour violation. In the scenario, $A_0, m_0^2, M_{1/2}$ are all related to a gravitino mass $m_{3/2}$.

As n_1 was only constrained to be between $-\sigma/2$ and 0, we allow it to vary within this range. We define the model at the GUT scale as:

$$m_0^2 = \frac{1}{4}m_{3/2}^2\,, \qquad A^0 = \sqrt{\frac{3}{4}}m_{3/2}\,, \qquad M_{1/2} = \sqrt{\frac{3}{4}}m_{3/2}\,. \tag{10.29}$$

This setup of the soft parameters corresponds to benchmark point A in [37]. The results are as follows, for Fit 1 the code being used can not generate any low energy data for this fit so we do not find any safe $B(\mu \to e\gamma)$ region using the conditions presented above. The Fit 2 has $\mathrm{BR}(\mu \to e\gamma) <= 10^{-30}$ which is unattainably low, thus this fit is plausible within the context of the minimal sugra conditions that have been specified. The smallness of the branching ratio for fit 2 comes about because with no RG running, in mSUGRA this rate would be exactly zero. The RG flavour violation will come from terms proportional to $Y^{\nu\dagger}Y^{\nu}$, whose elements are tiny (the largest is $O(10^{-14})$).

The Fit 3 generates a tachyonic s-electron for the full $(m_{3/2}, n_1)$ range. This is not to say that this fit will always have a tachyonic s-electron in other, less trivial embeddings.

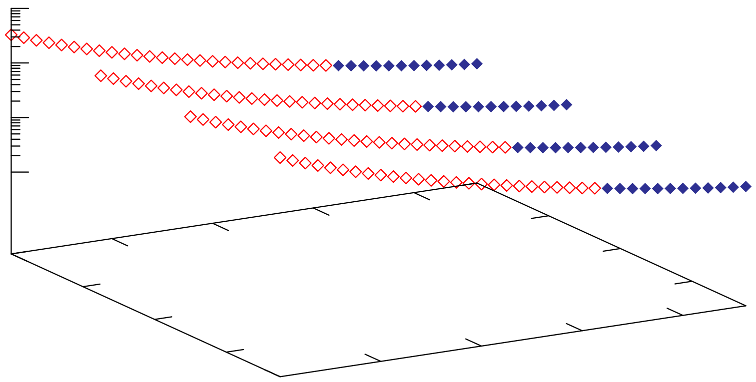

Figure 1: BR($\mu \to e\gamma$) for fit 4, with $\langle\Sigma\rangle = O(M_G)$. The solid points are below the experimental limit of $1.1 \cdot 10^{-11}$, and the hollow points are above.

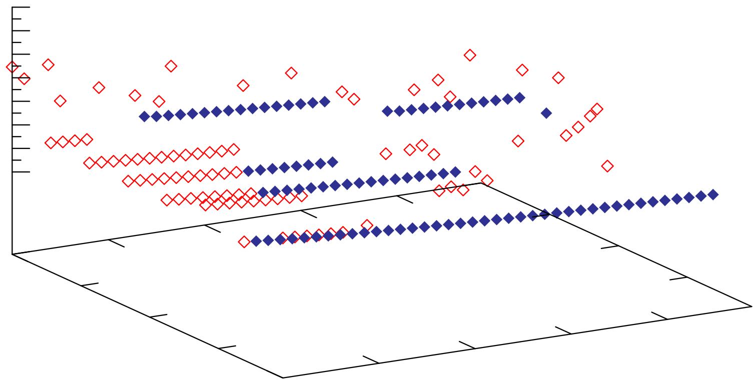

Figure 2: BR($\mu \to e\gamma$) for fit 4, with $\langle\Sigma\rangle = O(M_{Pl})$. The solid points are below the experimental limit of $1.1 \cdot 10^{-11}$, and the hollow points are above.

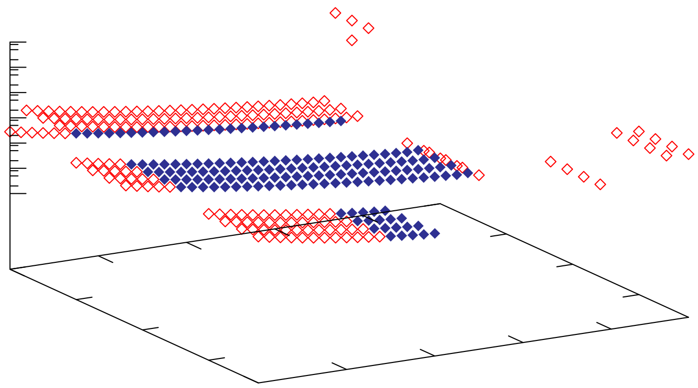

Figure 3: BR($\mu \to e\gamma$) for fit 5, with $\langle\Sigma\rangle = O(M_G)$. The solid points are below the experimental limit of $1.1 \cdot 10^{-11}$, and the hollow points are above.

Fits 4 and 5 produce regions below and above the experimental limits on $B(\mu \to e\gamma)$, the graphs for these fits appear in tables 1–4.

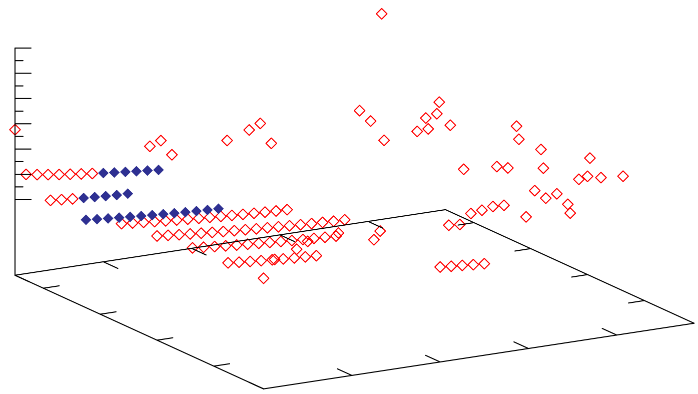

Figure 4: BR($\mu \to e\gamma$) for fit 5, with $\langle\Sigma\rangle = O(M_{Pl})$. The solid points are below the experimental limit of $1.1 \cdot 10^{-11}$, and the hollow points are above.

11. Conclusions

In summary, we began our analysis by reviewing the Green-Schwartz (GS) conditions for anomaly cancellation for theories based on a U(1) family symmetry. We then used these conditions to fix the charges of all the quark, lepton and Higgs fields and studied possibilities where the Higgs mass μ term is either present or absent in the original superpotential. The solutions which we constructed do not necessarily require an underlying Grand Unified Theory (GUT) but may be consistent with unification because of the GS conditions. Regardless of the presence of an explicit unified gauge group, the explicit solutions can produce matrices of the form that are identical to those that would be expected in an SU(5) case or Pati-Salam unified theory, for example.

The flavour structure of the resulting Yukawa matrices is controlled by the charges of the quarks and leptons under the U(1) family symmetry gauge group. We have determined the charges which are consistent with anomaly cancellation, and studied cases which can reproduce quark Yukawa matrices satisfying the Gatto-Sartori-Tonin (GST) relation, as well as other cases which do not satisfy the GST relation. We find the GST relation to be an appealing description of the value of the element V_{us}, and the GST relation provides a useful criterion for classifying flavour models. In our view, having the Cabibbo angle emerging automatically from a flavour model should have a similar status to gauge coupling unification in a high scale model. Having classified the solutions in terms of the GST condition, we then further classify the solutions according to which of them can produce the observed mixings in the lepton sector, and those that are consistent with a sub-class of solutions based on the SRHND or sequential dominance scenario with the further condition that the charges of the lepton doublets for the second and third family are equal, $l_2 = l_3$. We find that the GST solutions combined with SRHND results in highly fractional charges. On the other hand non-GST solutions with SRHND results in simpler charges, and we have therefore studied both sorts of examples.

We have presented three numerical examples of solutions satisfying the GST relation and two examples of non-GST solutions in order to compare how well these solutions fit the experimental information while maintaining $O(1)$ coefficients. For the GST solutions, one of these examples corresponds to a model that can be thought of as coming from an underlying SU(5) and for which a μ term is allowed in the superpotential. It is well known that in this case, given the relation $Y^e = Y^{d\,T}$, there should be a Clebsch-Gordan coefficient different in the charged lepton $(2,3)$ sector and in the $(2,3)$ d-quark sector in order to produce appropriate mixings in the context of the U(1) flavour symmetry and the GUT theory. Two other GST examples are presented for which the μ term is not allowed and which are not consistent with an underlying SU(5), or other GUT theory. In these cases $Y^e \neq (Y^d)^T$ but it is possible to maintain the relation $m_\tau \approx m_b$ and in one of them just the $O(1)$ coefficients of the underlying U(1) theory can account for the appropriate mixings in the charged lepton and d-quark sector. The non-GST cases also give a good description of masses and mixings, although in this case we need to rely on further coefficients, possible Clebsch-Gordan coefficients from an underlying GUT, in order to achieve a good phenomenological description.

For the above examples we have provided detailed numerical fits of the $O(1)$ coefficients required to reproduce the observed masses and mixings in both quark and lepton sectors. The purpose of performing such fits is to compare how well the different models can fit the data, and to try to determine quantitatively the best possible model corresponding to the best possible fit. Although in the cases just mentioned the solutions which fit the data best are the solutions consistent with an underlying SU(5) theory, the other two fits are quite plausible and represent interesting possibilities which cannot be excluded. Since all the models constructed have good agreement with the fermion masses and mixings, we clearly need further criteria in order to discriminate between the different classes of U(1) family symmetry models.

One may ask the more general question whether family symmetries based on abelian or non-abelian gauge groups are generically preferred? In order to address this question, we have extended the fit to include a generic symmetric form of quark and lepton mass matrices that can be understood in the context of a theory based on SU(3) family symmetry. We have found that overall the generic SU(3) family symmetry produces Yukawa matrices which tend to fit the data better, although the effect is not decisive, and one cannot draw a strong conclusion based solely on fits to fermion masses and mixings (or the way they can be reproduced). We have therefore enumerated some other possible criteria that are important in order to further discriminate among different flavour theories. Including the effects from the supersymmetric sector provides an additional way to discriminate among different theories based on their different predictions for soft masses and the resulting flavour changing processes and CP violation. We have presented two frameworks in which these processes can be studied in the context of flavour theories. The first is a non-minimal sugra scenario where family symmetries may render the Kähler metric diagonal at the flavour symmetry breaking scale, with off-diagonal elements arising only due to RG contributions and the non-degeneracy of soft masses. The second framework is a minimal sugra scenario for which a numerical exploration of $\mu \to e\,\gamma$ was performed. The results of this analysis shows marked differences between the different models presented. Of the GST cases only one survives the test of $B(\mu \to e\,\gamma)$ while for all of the non-GST cases presented there exist regions compatible with the $B(\mu \to e\,\gamma)$ experimental limit.

In conclusion, at the present time, phenomenological analyses provide some guidance about what family symmetry approaches may be valid, but do not yet allow one to draw any firm conclusion. More specific assumptions or data in the supersymmetric sector are needed in order to further discriminate between classes of models based on different family symmetry, unification or GST criteria.

Acknowledgments

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A. Conventions for the Yukawa diagonalization matrices

We diagonalize the Yukawa matrices, Y^f, with the unitary matrices V_L^f and V_R^f such that $Y^f_{\rm Diag} = V_L^{f\dagger} Y^f V_R^f$. V^f can be parameterized as

$$V^{f\dagger} = \begin{bmatrix} e^{i\alpha_1^f} & 0 & 0 \\ 0 & e^{i\alpha_2^f} & 0 \\ 0 & 0 & e^{i\alpha_3^f} \end{bmatrix} R_{12}^{fT} R_{13}^{fT} \begin{bmatrix} 1 & 0 & 0 \\ 0 & e^{i\beta_3^f} & 0 \\ 0 & 0 & 1 \end{bmatrix} R_{23}^{fT} \begin{bmatrix} 1 & 0 & 0 \\ 0 & e^{i\beta_2^f} & 0 \\ 0 & 0 & e^{i\beta_1^f} \end{bmatrix}, \tag{A.1}$$

where a plane R^f rotation has the form:

$$R_{23}^f = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c_{23}^f & -s_{23}^f \\ 0 & s_{23}^f & c_{23}^f \end{bmatrix}. \tag{A.2}$$

In this notation, the CKM matrix is $V = V_L^{u\dagger} V_L^d$.

B. Comparison to experimental information

The experimental information determining $V_{\rm CKM}$, usually put in terms of the Wolfenstein parameters A, λ, ρ and η, is extracted mainly from semileptonic decays of B mesons, CP violation in the K system, $B^0_{d,s} - \bar{B}^0_{d,s}$ oscillations, and CP asymmetries in various B decays. We use a fit, based in a bayesian approach (see for example [19] and [40]), of the the parameters A, λ, ρ and η including all the available information. Once we have done this, we compare the predictions of the mass textures with *the fitted parameters* because these include in a statistical way the experimental information from all the experiments considered. In the limit where we neglect all supersymmetric contributions to these observables, the fitted values for $\bar\rho$ and $\bar\eta$ are

$$\bar\rho = 0.199^{+0.053}_{-0.049}\,, \qquad \bar\eta = 0.328^{+0.037}_{-0.036}\,. \tag{B.1}$$

Parameter	Exp. value	Value at M_X
$\frac{V_{ub}}{V_{cb}}$	$(9.16 \pm 0.67) \times 10^{-2}$	
$\frac{V_{td}}{V_{ts}}$	0.1989 ± 0.0093	
V_{us}	0.224 ± 0.0036	
$\mathrm{Im}\{J\}$	$(2.88 \pm 0.4) \times 10^{-5}$	$(1.4 \pm 0.5) \times 10^{-5}$
$\frac{m_u}{m_c}$	$(1.9 \pm 0.19) \times 10^{-3}$	
$\frac{m_c}{m_t}$	$(7.5 \pm 1.7) \times 10^{-3}$	$(2.6 \pm 1.8) \times 10^{-3}$
$\frac{m_d}{m_s}$	$(5.2 \pm 0.35) \times 10^{-2}$	
$\frac{m_s}{m_b}$	$(6.4 \pm 2.3) \times 10^{-2}$	$(4.5 \pm 2.4) \times 10^{-2}$

Table 18: Experimental values used for the fit of numerical coefficients in the quark Yukawa matrices.

In order to compare to the V_{CKM} prediction, as given by the U(1) symmetries, we need to choose four elements, or combinations of them, we choose

$$\frac{|V_{ub}|}{|V_{cb}|}, \qquad \frac{|V_{td}|}{|V_{ts}|}, \qquad |V_{us}|, \qquad \mathrm{Im}\{J\}, \tag{B.2}$$

where J is the Jarlskog invariant. We choose these parameters because they can be put neatly in terms of the Wolfenstein parameters

$$\begin{aligned} \frac{|V_{ub}|}{|V_{cb}|} &= \frac{\lambda}{c_\lambda}\sqrt{\bar{\rho}^2+\bar{\eta}^2}\,, \qquad \frac{|V_{td}|}{|V_{ts}|} = \frac{\lambda}{c_\lambda}\sqrt{(c_\lambda-\bar{\rho})^2+\bar{\eta}^2}\,, \qquad |V_{us}| = \lambda\,, \\ \mathrm{Im}\{J\} &= A^2\lambda^6\bar{\eta}\,. \end{aligned} \tag{B.3}$$

To include the information of the quark masses we use

$$\frac{m_u}{m_c}, \qquad \frac{m_c}{m_t}, \qquad \frac{m_d}{m_s}, \qquad \frac{m_s}{m_b}. \tag{B.4}$$

The ratios $\frac{m_u}{m_c}$ and $\frac{m_d}{m_s}$ can be determined from the best measured ratios of the following mass ratios and the Q parameter, which is determined accurately from chiral perturbation theory;

$$\frac{m_u}{m_d}, \qquad \frac{m_c}{m_s}, \qquad Q = \frac{m_s/m_d}{\sqrt{1-(\frac{m_u}{m_d})^2}}. \tag{B.5}$$

We note here that a change in m_s with respect to previous similar fits [19] has an impact in the coefficients determined for the SU(3) symmetry, although it is consistent with previous determinations if we consider the errors involved. We have used here $m_c/m_s = 15.5 \pm 3.7$ in contrast to $m_c/m_s = 9.5 \pm 1.7$ as used in [19]. We put in table 18 the experimental values of the parameters that we use to determine the coefficients of the Yukawa texture. From equations eq. (B.2) and eq. (B.4) we see that we can fit only *eight* parameters of both of the Yukawa matrices as given by the U(1) symmetries, but we will see that in most cases that is sufficient in order to account for the viability of a given ansatz or symmetry for the matrices. We also need to take into account the RGE effects in going from the electroweak scale to the scale at which the U(1) symmetry breaks. We assume first that

Parameter	Exp. value
$\tan\theta_{23}$	1.07 ± 0.37
$\tan\theta_{13}$	0.21 ± 0.1 (u.b)
$\tan\theta_{12}$	0.65 ± 0.12
m_{ν_3}	0.05 ± 0.01
$\frac{m_{\nu_2}}{m_{\nu_3}}$	0.19 ± 0.05

Table 19: Experimental values [43] used for the fit of numerical coefficients in the neutrino Yukawa matrix. For $\tan\theta_{13}$ we have fitted using the upper bound.

this is the GUT scale, so in order to determine the values of the parameters defining the U(1) symmetry (analogously for the SU(3) case) we take the values of the parameters appearing in eqs. (B.2)–(B.4) at the GUT scale. One of the reasons for using mass ratios instead of just masses is because the RGE effects on the mass ratios has less impact than for the masses. This fit of the parameters defining the U(1) symmetry is performed with the aid of the MINUIT package adapted for root [42]. In this way we are able to compare how well a symmetry is fitted to the experimental information, and compare among the fits for different symmetries.

We do a completely analogous analysis in the neutrino sector, using the following observables

$$\tan\theta_{23}\,, \qquad \tan\theta_{13}\,, \qquad \tan\theta_{12}\,, \qquad m_{\nu_3}\,, \qquad \frac{m_{\nu_2}}{m_{\nu_3}} \tag{B.6}$$

and their experimental values as appear in table 19

B.1 Evaluation of observables with fitted parameters

In this section we put the evaluation, at the scale M_X, of the experimental inputs using the fitted parameters in order to compare with table 18.

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	GST sol. 2 $u=v=0$	GST sol. 2 $u\neq v\neq 0$	GST sol. 3 $u\neq v\neq 0$
Parameter	Evaluation at M_X		
$\frac{V_{ub}}{V_{cb}}$	$(9.15\pm 0.52)\times 10^{-2}$	$(9.12\pm 0.35)\times 10^{-2}$	$(9.31\pm 0.27)\times 10^{-2}$
$\frac{V_{td}}{V_{ts}}$	0.199 ± 0.081	0.199 ± 0.031	0.20 ± 0.078
V_{us}	0.225 ± 0.0041	0.225 ± 0.0043	0.225 ± 0.0040
$\mathrm{Im}\{J\}$	$(1.4\pm 0.3)\times 10^{-5}$	$(1.4\pm 0.3)\times 10^{-5}$	$(0.8\pm 0.6)\times 10^{-5}$
$\frac{m_u}{m_c}$	$(1.9\pm 0.21)\times 10^{-3}$	$(1.9\pm 0.16)\times 10^{-3}$	$(1.8\pm 0.21)\times 10^{-3}$
$\frac{m_c}{m_t}$	$(2.1\pm 0.73)\times 10^{-3}$	$(1.9\pm 0.47)\times 10^{-3}$	$(4.\pm 1.54)\times 10^{-3}$
$\frac{m_d}{m_s}$	$(4.9\pm 0.5)\times 10^{-2}$	$(4.9\pm 0.5)\times 10^{-2}$	$(4.9\pm 0.6)\times 10^{-2}$
$\frac{m_s}{m_b}$	$(3.0\pm 1.7)\times 10^{-2}$	$(1.9\pm 2.1)\times 10^{-2}$	$(4.4\pm 1.3)\times 10^{-2}$
$\tan\theta_{23}$	1.27 ± 0.39	1.02 ± 0.41	1.00 ± 0.42
$\tan\theta_{13}$	0.21 ± 0.32	0.21 ± 0.39	0.27 ± 0.19
$\tan\theta_{12}$	0.65 ± 0.41	0.65 ± 0.42	0.65 ± 0.32
m_{ν_3}	0.046 ± 0.05	0.023 ± 0.05	0.02 ± 0.05
$\frac{m_{\nu_2}}{m_{\nu_3}}$	0.19 ± 0.05	0.16 ± 0.08	0.19 ± 0.05
	Non GST sol. 1 $u=v=0$	Non GST sol. 2 $u=v=0$	
Parameter	Evaluation at M_X		
$\frac{V_{ub}}{V_{cb}}$	$(9.59\pm 0.49)\times 10^{-2}$	$(9.14\pm 0.65)\times 10^{-2}$	
$\frac{V_{td}}{V_{ts}}$	0.201 ± 0.073	0.199 ± 0.033	
V_{us}	0.225 ± 0.0037	0.225 ± 0.0027	
$\mathrm{Im}\{J\}$	$(1.7\pm 0.4)\times 10^{-5}$	$(1.4\pm 0.3)\times 10^{-5}$	
$\frac{m_u}{m_c}$	$(1.9\pm 0.33)\times 10^{-3}$	$(1.8\pm 0.15)\times 10^{-3}$	
$\frac{m_c}{m_t}$	$(1.1\pm 0.82)\times 10^{-3}$	$(2.5\pm 0.56)\times 10^{-3}$	
$\frac{m_d}{m_s}$	$(5.2\pm 0.3)\times 10^{-2}$	$(4.9\pm 0.5)\times 10^{-2}$	
$\frac{m_s}{m_b}$	$(3.2\pm 1.2)\times 10^{-2}$	$(4.5\pm 2.2)\times 10^{-2}$	
$\tan\theta_{23}$	0.67 ± 0.41		
$\tan\theta_{13}$	0.19 ± 0.16		
$\tan\theta_{12}$	0.55 ± 0.31		
m_{ν_3}	0.038 ± 0.04		
$\frac{m_{\nu_2}}{m_{\nu_3}}$	0.13 ± 0.06		

Table 20: Evaluated values of experimental observables using the fitted parameters for the SU(5) GST case, for the $u\neq -v\neq 0$ cases, the SU(5) non GST case and the SU(3) like cases considered in section 9.

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	SU(3), $\Phi_1 < 0$	SU(3), $\Phi_1 > 0$
Parameter	Evaluation at M_X	
$\frac{V_{ub}}{V_{cb}}$	$(9.14 \pm 0.65) \times 10^{-2}$	$9.13 \pm 0.21) \times 10^{-2}$
$\frac{V_{td}}{V_{ts}}$	0.199 ± 0.023	0.198 ± 0.024
V_{us}	0.225 ± 0.0027	0.225 ± 0.0033
$\mathrm{Im}\{J\}$	$(1.4 \pm 0.3) \times 10^{-5}$	$(1.4 \pm 0.3) \times 10^{-5}$
$\frac{m_u}{m_c}$	$(1.8 \pm 0.15) \times 10^{-3}$	$(1.9 \pm 0.15) \times 10^{-3}$
$\frac{m_c}{m_t}$	$(2.5 \pm 0.56) \times 10^{-3}$	$(2.5 \pm 0.56) \times 10^{-3}$
$\frac{m_d}{m_s}$	$(4.9 \pm 0.5) \times 10^{-2}$	$(4.9 \pm 0.5) \times 10^{-2}$
$\frac{m_s}{m_b}$	$(4.5 \pm 2.2) \times 10^{-2}$	$(4.5 \pm 2.2) \times 10^{-2}$

Table 21: Evaluated values of experimental observables using the fitted parameters for the SU(3) like cases considered.

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